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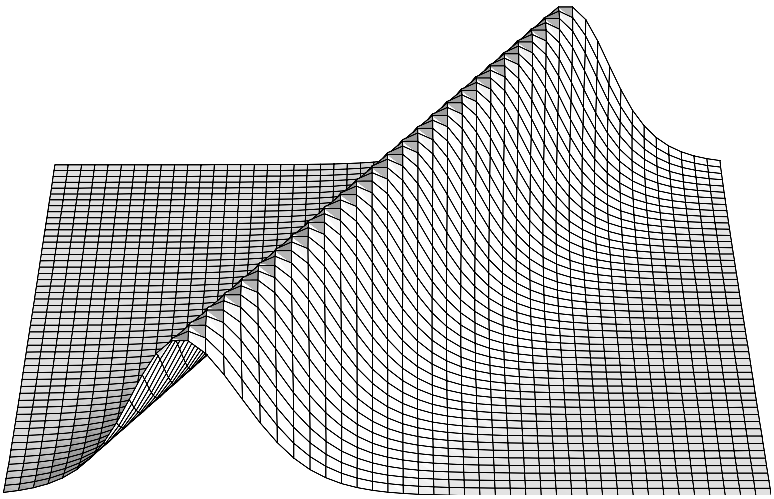
Bilinearization of soliton equations



One-soliton solution.

1.1 Solitary waves and solitons

The word 'wave' normally makes us think of a wave train as shown in Figure 1.1. However, when surfing off a gently sloping beach, we often make use of a *solitary wave* (see Figure 1.2). A *soliton* is a type of solitary wave which maintains its identity after it collides with another wave of the same kind. Let us first study wave equations which describe solitary waves.

A wave equation having soliton solutions has both nonlinearity and dispersion. Before studying how to solve such a wave equation, we will investigate

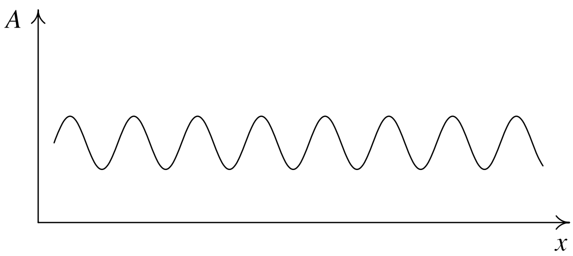


Figure 1.1. A wave train. Amplitude A, position x.

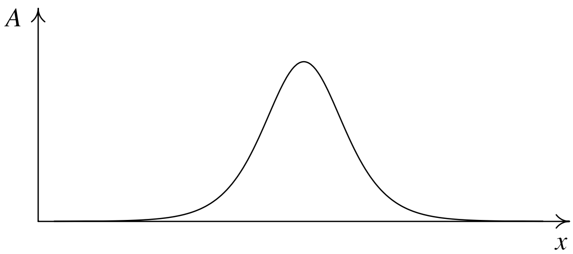


Figure 1.2. A solitary wave.

the influence that nonlinearity and dispersion have on the behaviour of a wave. We will also try to understand, using intuitive arguments, under what conditions a solitary wave can exist.

1.2 Nonlinearity and dispersion

1.2.1 Linear nondispersive waves

Typical examples of the simplest kind of waves are sound waves and electromagnetic waves. They are governed by

$$\left(\frac{\partial^2}{\partial t^2} - v_0^2 \frac{\partial^2}{\partial x^2}\right) f(x,t) = 0, \tag{1.1}$$

where v_0 is a constant representing the wave speed. Since this equation can be formally decomposed as

$$\left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x}\right) f(x,t) = 0,$$

let us consider the simpler form,

$$\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x}\right) f(x,t) = 0. \tag{1.2}$$

A solution of this equation also satisfies (1.1). While (1.1) gives travelling wave solutions moving to the left and to the right, (1.2) gives only the right-moving ones,

$$f(x,t) = f(x - v_0 t).$$

Assuming that this wave is periodic, the most fundamental solution is the plane wave,

$$f(x,t) = \exp[\mathrm{i}(\omega t - kx)].$$

The relationship between the angular frequency ω and the wave number k is given by $\omega = v_0 k$, where the constant v_0 is the phase velocity of the wave. This is called the *dispersion relation* and, in this case, it is linear.

A wave governed by a linear dispersion relation is called a *nondispersive wave*. A feature of such a wave is that an initial profile taking the form of a pulse, which is made up of a superposition of plane waves with different wave numbers k, does not change its shape. This is because each of the superposed plane waves travels with the same speed. Waves with unchanging shape play a very important role in applications as a means of communication. A soliton, even though it is not a nondispersive wave, possesses the above property of unchanging shape and, because of this, it should have practical applications.

Next, we will investigate a particular linear dispersive wave equation.

1.2.2 Linear dispersive waves

We consider, as the simplest example, the wave equation

$$\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x} + \delta \frac{\partial^3}{\partial x^3}\right) f(x,t) = 0. \tag{1.3}$$

If we suppose that it has a plane wave solution

$$f(x,t) \propto \exp[\mathrm{i}(\omega t - kx)],$$

then the dispersion relation is given by

$$\omega = v_0 k - \delta k^3,$$

which is nonlinear with respect to k. Hence, the phase velocity is different from that in Section 1.2.1 and is given by

$$\frac{\omega}{k} = v_0 - \delta k^2,$$

which depends on a wave number k. On the other hand, its group velocity is given by

$$\frac{\partial \omega}{\partial k} = v_0 - 3\delta k^2.$$

We remark that, if $\delta > 0$, both velocities are less than v_0. Since the velocity of each of the plane waves which make up an initial wave vary with k, the wave spreads out as it travels. This shows that linear dispersive waves do not preserve their original shape.

The two examples we have discussed so far are both linear differential equations. Next we consider the influence of nonlinearity.

1.2.3 Nonlinear nondispersive waves

We consider, as the simplest example, the nondispersive wave equation,

$$\left(\frac{\partial}{\partial t} + v(f)\frac{\partial}{\partial x}\right) f(x,t) = 0, \tag{1.4}$$

where $v(f) = v_0 + \alpha f^m$. This equation is a nonlinear wave equation in which the speed $v(f)$ depends on the amplitude f.

Equation (1.4) has the formal solution

$$f(x,t) = f(x - v(f)t),$$

and if $v = v(f)$ is an increasing function in f, this formula tells us that a wave travels faster as its amplitude increases. Therefore, as one can see from Figure 1.3, the wave steepens and then breaks. Physically speaking, however, before the wave breaks, its gradient $|\partial f/\partial x| \gg 1$. When this happens, (1.4) becomes meaningless and must be replaced by the differential equation (1.5) presented in Section 1.2.4.

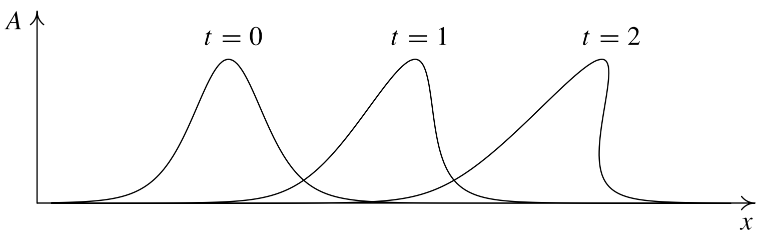


Figure 1.3. Steepening of a solitary wave. A wave which is symmetrical at $t = 0$ steepens and breaks because of the dependence of the wave speed on its amplitude.

1.2.4 Nonlinear dispersive waves

The equation

$$\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x} + \alpha f^m \frac{\partial}{\partial x} + \delta \frac{\partial^3}{\partial x^3}\right) f(x, t) = 0 \tag{1.5}$$

possesses a pulse-like wave solution which travels with unchanging shape; a solitary wave solution. Before finding this solution by mathematical means, let us look into the physical reasons for the existence of such a solution. We have seen in Sections 1.2.2 and 1.2.3 that neither a linear dispersive solitary wave nor a nonlinear nondispersive solitary wave can exist. Why then can a solitary wave solution exist for a wave equation which has both nonlinearity and dispersion?

Assuming that the solitary wave shown in Figure 1.4 exists, let us investigate whether this pulse-like wave can travel with unchanging shape. To this end it is necessary, at least, that the velocities at the top and base of the wave have the same value v. In order to investigate further, we introduce a new space coordinate $\eta = px - \Omega t$, where $v = \Omega/p$ and p is a free parameter. The parameter p is such that, as it increases, the pulse becomes sharper. It is convenient to introduce η to describe a wave which travels at a constant speed v. For $t = 0$, η is proportional to x ($\eta = px$), and, for $t \neq 0$, $\eta = p(x - vt)$, being proportional to $x - vt$, travels at a speed v.

If the maximum of the wave amplitude f is A, occurring at $\eta = 0$, then in a neighbourhood of this point we have

$$f \sim A(1 - \text{constant} \times \eta^2),$$

because here the height f can be approximated by a quadratic expression in η. From this equation, we have $\partial^3 f/\partial x^3 \sim 0$ and therefore, in the neighbourhood

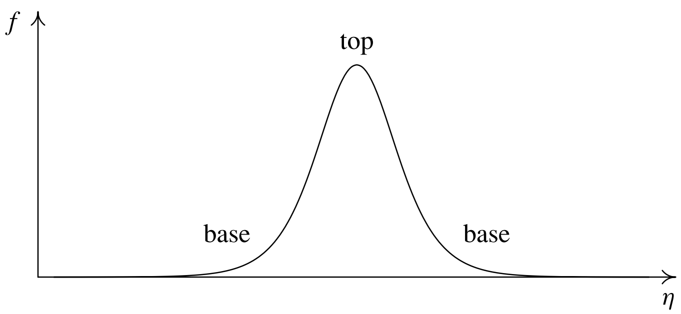


Figure 1.4. Splitting a solitary wave into its top and base.

of the top of the wave, f satisfies the differential equation

$$\left(\frac{\partial}{\partial t} + [v_0 + \alpha f(x,t)^m]\frac{\partial}{\partial x}\right) f(x,t) \sim 0.$$

This equation is the same as (1.4) and so, as described in Section 1.2.3, the speed at the top of the wave, at which the amplitude is A, is given by

$$v(f) = v_0 + \alpha A^m. \tag{1.6}$$

From this it is clear that $v(f)$ is larger than v_0 if $\alpha > 0$.

On the other hand, at the base of the wave, we can neglect the nonlinear term because f is very small, and so f satisfies the linear differential equation

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x} + \delta\frac{\partial^3}{\partial x^3}\right) f(x,t) \sim 0. \tag{1.7}$$

As seen in Section 1.2.2, the group and phase velocities at the base, v_{gr} and v_{pf}, respectively, are given by

$$\begin{aligned} v_{gr} &= \frac{\partial \omega}{\partial k} = v_0 - 3\delta k^2, \\ v_{pf} &= \frac{\omega}{k} = v_0 - \delta k^2. \end{aligned} \tag{1.8}$$

From this we see that both the group and phase velocities are smaller than v_0 if $\delta > 0$, and so the speed at the top of a solitary wave is larger than that at the base. This indicates that the original shape of the wave is not preserved, and a solitary wave cannot exist. This disagrees with the experimental observation. What is wrong with the above argument?

In fact, our calculation of the velocity at the base of the wave is incorrect. Since the amplitude at the base is small, the differential equation is certainly linear. The error arose because we approximated the solution as a plane wave and considered the wave velocity to be the linear (phase or group) velocity, according to the common understanding of linear waves. The base of the wave is not made up of a superposition of linear plane waves

$$f \sim \exp[\pm \mathrm{i}(kx - \omega t)]$$

or

$$f \sim \sin(kx - \omega t),$$

but, in fact, is expressed in terms of exponentially decaying waves

$$f \sim \exp[\pm(px - \Omega t)]. \tag{1.9}$$

Since these expressions for f tend to infinity as $x \to \infty$ or $x \to -\infty$, and so do not satisfy a physical boundary condition, solutions of this form are

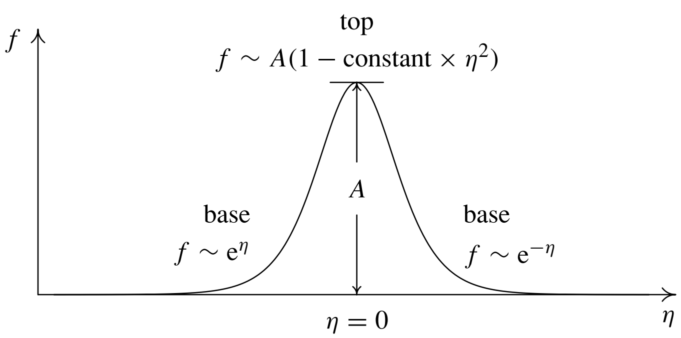


Figure 1.5. Approximation of a solitary wave at its top and base.

normally discarded as physically meaningless in the theory of linear waves. In the theory of nonlinear waves, however, we can construct a global solution by connecting local solutions, as illustrated in Figure 1.5.

The base of the wave is now expressed in terms of exponentially decaying solutions $f \sim \exp(\pm\eta)$ and, from (1.7), we obtain the relationship, called the *nonlinear dispersion relation*,

$$\Omega = v_0 p + \delta p^3.$$

The wave velocity v is then given by

$$v = \frac{\Omega}{p} = v_0 + \delta p^2, \tag{1.10}$$

which coincides with the velocity at the top (1.6) if and only if

$$\delta p^2 = \alpha A^m.$$

If p and A satisfy this equation, then the solitary wave can travel without changing its shape. Recall that p is the parameter associated with the width of the pulse; as p increases, the pulse becomes steeper and narrower. This formula (if $\delta, \alpha, m > 0$) therefore indicates that as the amplitude of the pulse increases, it becomes sharper.

The above discussion suggests an important idea for solving nonlinear wave equations. When trying to obtain a solution by a perturbation method, we cannot employ, as a first approximation, the normal plane wave solutions $f \sim \sin(kx - \omega t)$ but should instead use the exponentially decaying solutions $f \sim \exp[\pm(px - \Omega t)]$, which are rejected in linear wave theory. More precisely, we expand f into a power series in $\varepsilon \exp(\eta)$ as

$$f(x,t) \sim \varepsilon a_1 \exp(\eta) + \varepsilon^2 a_2 \exp(2\eta) + \cdots, \tag{1.11}$$

where $\eta = px - \Omega t$ and ε is a small parameter.

However, if η is sufficiently large, we have seen (see Figure 1.5) that

$$f(x,t) \sim \exp(-\eta). \tag{1.12}$$

Expanding f into a power series of $\exp(\eta)$ and finding a solution asymptotic to $\exp(-\eta)$ as $\eta \to +\infty$ corresponds to finding a Padé approximation for f,

$$f = G/F. \tag{1.13}$$

If this correspondence is correct, the fundamental idea of the direct method, referred to in the title of this book, is to find solutions of nonlinear differential equations through dependent variable transformations like $f = G/F$.

Remark

For a function f with formal power series

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots, \tag{1.14}$$

the method of Padé approximation [1] expresses $f(x)$ as a ratio of polynomials G and F. This gives an approximate analytic continuation for $f(x)$ that can be used to obtain information for large x.

For example, the power series

$$f(x) = x - x^3 + x^5 - x^7 + \cdots \tag{1.15}$$

converges to a finite value in a region $|x| < 1$ and therefore gives properties of $f(x)$ in this region. In the region $|x| \geq 1$, however, this is a divergent series and so it does not make sense to use it, for example, to find the value at $x = 2$. If we express $f(x)$ as the rational function

$$f(x) = \frac{x}{1+x^2},$$

then $f(x) \sim x^{-1}$ for $|x| \gg 1$. In particular, the substitution of $x = \exp(\eta)$ yields

$$\begin{aligned} f(x) &= \exp(\eta) - \exp(3\eta) + \exp(5\eta) - \cdots \\ &= \frac{\exp(\eta)}{1+\exp(2\eta)} \\ &\sim \exp(-\eta) \quad (\eta \gg 1) \end{aligned} \tag{1.16}$$

and gives the correct behaviour of $f(x)$, even though η is large. □

Here we have used an intuitive argument to find general properties of the solitary wave solution without any knowledge of the precise form of the nonlinear term. In Section 1.3 we will investigate to what extent this solution coincides with the exact solution.

1.3 Solutions of nonlinear differential equations

In Section 1.2 we discussed solutions of the nonlinear differential equation

$$\left(\frac{\partial}{\partial t'} + v_0\frac{\partial}{\partial x'} + \alpha f^m \frac{\partial}{\partial x'} + \delta\frac{\partial^3}{\partial x'^3}\right) f(x', t') = 0, \tag{1.17}$$

using an intuitive argument (we use variables x', t' for later convenience). Let us here investigate the properties of such solutions by mathematical means.

First, we consider the independent variable transformation

$$\begin{aligned} x &= x' - v_0 t', \\ t &= t', \end{aligned} \tag{1.18}$$

describing a frame moving at velocity v_0. Under this transformation, partial derivatives become

$$\begin{aligned} \frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} - v_0\frac{\partial}{\partial x}, \\ \frac{\partial}{\partial x'} &= \frac{\partial}{\partial x}. \end{aligned} \tag{1.19}$$

In the moving frame, the second term in (1.17) is eliminated and so we obtain

$$\left(\frac{\partial}{\partial t} + \alpha f^m \frac{\partial}{\partial x} + \delta\frac{\partial^3}{\partial x^3}\right) f(x, t) = 0. \tag{1.20}$$

Next we make use of a *similarity transformation*. Under the scaling transformation $t = \varepsilon^3\tau$, $x = \varepsilon\xi$, where ε is a constant, (1.20) is equivalent to

$$\left(\frac{\partial}{\partial \tau} + \varepsilon^2\alpha f^m \frac{\partial}{\partial \xi} + \delta\frac{\partial^3}{\partial \xi^3}\right) f(\varepsilon\xi, \varepsilon^3\tau) = 0, \tag{1.21}$$

and then the dependent variable transformation $f(\varepsilon\xi, \varepsilon^3\tau) = \varepsilon^{-2/m} f'(\xi, \tau)$ yields

$$\left(\frac{\partial}{\partial \tau} + \alpha (f')^m \frac{\partial}{\partial \xi} + \delta\frac{\partial^3}{\partial \xi^3}\right) f'(\xi, \tau) = 0. \tag{1.22}$$

This shows that if we replace f in (1.20) by

$$f'(\xi, \tau) = \varepsilon^{2/m} f(\varepsilon\xi, \varepsilon^3\tau), \tag{1.23}$$

then f' again satisfies the same differential equation. This is called a similarity transformation.

If we have a travelling wave solution $f = f(x - vt)$ then $\varepsilon^{2/m} f(x - \varepsilon^2 vt)$ will also be a solution. From this we see that if the amplitude of a solitary wave increases by a factor $\varepsilon^{2/m}$ then its velocity v increases by a factor of ε^2. This shows that, even in the case that an exact solution cannot be found, we can still determine some properties of these solutions by their similarity transformations. In fact, exact analytic solutions for few nonlinear differential equations are known.

The relation between the speed and amplitude of solitary waves can also be found directly from the partial differential equation without employing a similarity transformation. Let us again consider the nonlinear partial differential equation

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x} + \alpha f^m \frac{\partial}{\partial x} + \delta\frac{\partial^3}{\partial x^3}\right) f(x,t) = 0. \tag{1.24}$$

We consider a solitary wave solution $f = f(x - vt)$ travelling at constant speed v. Then we have

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x}, \tag{1.25}$$

and (1.24) reduces to

$$\left((-v + v_0)\frac{\partial}{\partial x} + \alpha f^m \frac{\partial}{\partial x} + \delta\frac{\partial^3}{\partial x^3}\right) f(x - vt) = 0. \tag{1.26}$$

Integrating the above equation with respect to x and using the boundary condition for solitary waves,

$$\frac{\partial^n}{\partial x^n} f(x - vt) \to 0 \quad (x \to \pm\infty) \quad n = 0, 1, 2, \ldots, \tag{1.27}$$

we have

$$(-v + v_0) f + \frac{\alpha}{m+1} f^{m+1} + \delta\frac{\partial^2}{\partial x^2} f = 0. \tag{1.28}$$

Multiplying by f_x on both sides and integrating with respect to x again, we obtain

$$(-v + v_0) f^2 + \frac{2\alpha}{(m+1)(m+2)} f^{m+2} + \delta f_x^2 = 0. \tag{1.29}$$

At the top of the solitary wave we suppose that $f = f_{\max}$, and we have

$$f_x = 0, \tag{1.30}$$