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Introduction

Like the simple additive structure on an Euclidean space, the more complicated algebraic structure on a Lie group provides a convenient setting under which various stochastic processes with interesting properties may be defined and studied. An important class of such processes are Lévy processes that possess translation invariant distributions. Since a Lie group is in general noncommutative, there are two different types of Lévy processes, left and right Lévy processes, defined respectively by the left and right translations. Because the two are in natural duality, for most purposes, it suffices to study only one of them and derive the results for the other process by a simple transformation. However, the two processes play different roles in applications. Note that a Lévy process may also be characterized as a process that possesses independent and stationary increments.

The theory of Lévy processes in Lie groups is not merely an extension of the theory of Lévy processes in Euclidean spaces. Because of the unique structures possessed by the noncommutative Lie groups, these processes exhibit certain interesting properties that are not present for their counterparts in Euclidean spaces. These properties reveal a deep connection between the behavior of the stochastic processes and the underlying algebraic and geometric structures of the Lie groups.

The study of Lie group–valued processes can be traced to F. Perrin's work in 1928. Itô's article on Brownian motions in Lie groups was published in 1951. Hunt [32] in 1956 obtained an explicit formula for the generator of a continuous convolution semigroup of probability measures that provides a complete characterization, in the sense of distribution, of Lévy processes in a general Lie group. More recently, Applebaum and Kunita [3] in 1993 proved that a Lévy process is the unique solution of a stochastic integral equation driven by a Brownian motion and a Poisson random measure. This corresponds to the Lévy–Itô representation in the Euclidean case and provides a pathwise characterization of Lévy processes in Lie groups.

The celebrated Lévy–Khintchine formula provides a useful Fourier transform characterization of infinite divisible laws, or distributions of Lévy processes, on Euclidean spaces. For Lévy processes in Lie groups, there is also a natural connection with Fourier analysis. Gangolli [22] in 1964 obtained a type of Lévy–Khintchine formula for spherically symmetric infinite divisible laws on symmetric spaces, which may be regarded as a result on Lévy processes in semi-simple Lie groups. More generally, Fourier methods were applied to study the probability measures on locally compact groups (see, for example, Heyer [28] and Siebert [55]) and to study the distributional convergence of random walks to Haar measures on finite groups and in some special cases Lie groups (see Diaconis [15] and Rosenthal [53]). Very recently, the author [43] studied the Fourier expansions of the distribution densities of Lévy processes in compact Lie groups based on the Peter–Weyl theorem and used it to obtain the exponential convergence of the distributions to Haar measures.

Perhaps one of the deepest discoveries in probability theory in connection with Lie groups is the limiting properties of Brownian motions and random walks in semi-simple Lie groups of noncompact type, both of which are examples of Lévy processes. Dynkin [16] in 1961 studied the Brownian motion x_t in the space X of Hermitian matrices of unit determinant. He found that, as time $t \to \infty$, x_t converges to infinity only along certain directions and at nonrandom exponential rates. The space X may be regarded as the homogeneous space $SL(d, \mathbb{C})/SU(d)$, where $SL(d, \mathbb{C})$ is the group of $d \times d$ complex matrices of determinant 1 and $SU(d)$ is the subgroup of unitary matrices. The Brownian motion x_t in X may be obtained as the natural projection of a continuous Lévy process in $SL(d, \mathbb{C})$. Therefore, the study of x_t in X may be reduced to that of a Lévy process in the Lie group $SL(d, \mathbb{C})$.

The matrix group $SL(d, \mathbb{C})$ belongs to a class of Lie groups called semi-simple Lie groups of noncompact type and the space X is an example of a symmetric space. Such a symmetric space possesses a polar decomposition under which any point can be represented by "radial" and "angular" components (both of which may be multidimensional). Dynkin's result says that the Brownian motion in X has a limiting angular component and its radial component converges to ∞ at nonrandom exponential rates. This result is reminiscent of the fact that a Brownian motion in a Riemannian manifold of pinched negative curvature has a limiting "angle" (see Prat [49]). However, the existence of the angular limit is not implied by this fact because a symmetric space of noncompact type may have sections of zero curvature.

Part of Dynkin's result, the convergence of the angular component, was extended by Orihara [47] in 1970 to Brownian motion in a general symmetric space of noncompact type. Malliavin and Malliavin [44] in 1972 obtained a complete result for the limiting properties of a horizontal diffusion in a general semi-simple Lie group of noncompact type. See also Norris, Rogers, and Williams [46], Taylor [56, 57], Babillot [6], and Liao [38] for some more recent studies of this problem from different perspectives.

In a different direction, Furstenberg and Kesten [21] in 1960 initiated the study of limiting properties of products of iid (independent and identically distributed) matrix or Lie group–valued random variables. Such processes may be regarded as random walks or discrete-time Lévy processes in Lie groups. The study of these processes was continued in Furstenberg [20], Tutubalin [58], Virtser [59], and Raugi [51]. In Guivarc'h and Raugi [24], the limiting properties of random walks on semi-simple Lie groups of noncompact type were established under a very general condition. These methods could be extended to a general Lévy process. This extension was made in Liao [40] and was applied to study the asymptotic stability of Lévy processes in Lie groups viewed as stochastic flows on certain homogeneous spaces.

The discrete-time random walks in Lie groups exhibit the same type of limiting properties as the more general Lévy processes. However, the limiting properties of continuous-time Lévy processes do not follow directly from them. The Lévy processes also include diffusion processes in Lie groups that possess translation invariant distributions. The limiting properties of such processes can be studied in connection with their infinitesimal characterizations, namely, their infinitesimal generators or the vector fields in the stochastic differential equations that drive the processes. For example, the limiting exponential rates mentioned here may be expressed in terms of the generator or the vector fields, which allows explicit evaluation in some special cases.

The limiting properties of Lévy processes may also be studied from a dynamic point of view. If the Lie group G acts on a manifold M, then a right Lévy process g_t in G may be regarded as a random dynamical system, or a stochastic flow, on M. The limiting properties of g_t as a Lévy process imply interesting ergodic and dynamical behaviors of the stochastic flow. For example, the Lyapunov exponents, which are the nonrandom limiting exponential rates at which the tangent vectors on M are contracted by the stochastic flow, and the stable manifolds, which are random submanifolds of M contracted by the stochastic flow at the fixed exponential rates, can all be determined explicitly in terms of the group structure.

The purpose of this work is to provide an introduction to Lévy processes in general Lie groups and, hopefully, an accessible account on the limiting and dynamical properties of Lévy processes in semi-simple Lie groups of noncompact type. The reader is assumed to be familiar with Lie groups and stochastic analysis. The basic definitions and facts in these subject areas that will be needed are reviewed in the two appendices at the end of this work. However, no prior knowledge of semi-simple Lie groups will be assumed.

Because the present work is not intended as a comprehensive treatment of Lévy processes in Lie groups, many interesting and related topics are not mentioned, some of which may be found in the survey article by Applebaum [1]. The distribution theory and limiting theorems of convolution products of probability measures in more general topological groups and semigroups can be found in Heyer [28] and in Högnäs and Mukherjea [29]. The theory of Lévy processes in Euclidean spaces has been well developed and there remains enormous interest in these processes (see Bertoin [9], Sato [54], and the proceedings in which [1] appears).

This work is organized as follows: The first chapter contains the basic definitions and results for Lévy processes in a general Lie group, including the generators of Lévy processes regarded as Markov processes, the Lévy measures that are the counting measures of the jumps of Lévy processes, and the stochastic integral equations satisfied by Lévy processes. In Chapter 2, we study the processes in homogeneous spaces induced by Lévy processes in Lie groups as one-point motions, we will discuss the Markov property of these processes and establish some basic relations among various invariance properties, and we will show that a Markov process in a manifold invariant under the action of a Lie group G is the one-point motion of a Lévy process in G. We will also discuss Riemannian Brownian motions in Lie groups and homogeneous spaces. Chapter 3 contains the proofs of some basic results stated in the previous two chapters, including Hunt's result on the generator [32] and the stochastic integral equation characterization of Lévy processes due to Applebaum and Kunita [3]. In Chapter 4, we study the Fourier expansion of the distribution densities of Lévy processes in compact Lie groups based on the Peter–Weyl theorem and obtain the exponential convergence of the distribution to the normalized Haar measure. The results of this chapter are taken from Liao [43]. In the first four chapters, only a basic knowledge of Lie groups will be required.

In the second half of the book, we will concentrate on limiting and dynamical properties of Lévy processes in semi-simple Lie groups of noncompact type. Chapter 5 provides a self-contained introduction to semi-simple Lie groups of noncompact type as necessary preparation for the next three chapters. The basic limiting properties of Lévy processes in such a Lie group are established in Chapters 6. To establish these results, we follow closely the basic ideas in Guivarc'h and Raugi [24], but much more detail is provided with considerable modifications. We also include in this chapter a simple and elementary proof of the limiting properties for certain continuous Lévy processes. Additional limiting properties under an integrability condition, including the existence of nonrandom exponential rates of the "radial components," are established in Chapter 7. In Chapter 8, the dynamical aspects of the Lévy processes are considered by viewing Lévy processes as stochastic flows on certain compact homogeneous spaces. We will obtain explicit expressions for the Lyapunov exponents, the associated stable manifolds, and a clustering pattern of the stochastic flows in terms of the group structures. The main results of this chapter are taken from Liao [40, 41, 42].

1

Lévy Processes in Lie Groups

This chapter contains an introduction to Lévy processes in a general Lie group. The left and right Lévy processes in a topological group G are defined in Section 1.1. They can be constructed from a convolution semigroup of probability measures on G and are Markov processes with left or right invariant Feller transition semigroups. In the next two sections, we introduce Hunt's theorem for the generator of a Lévy process in a Lie group G and prove some related results for the Lévy measure determined by the jumps of the process. In Section 1.4, the Lévy process is characterized as a solution of a stochastic integral equation driven by a Brownian motion and an independent Poisson random measure whose characteristic measure is the Lévy measure. Some variations and extensions of this stochastic integral equation are discussed. The proofs of the stochastic integral equation characterization, due to Applebaum and Kunita, and of Hunt's theorem, will be given in Chapter 3. For Lévy processes in matrix groups, a more explicit stochastic integral equation, written in matrix form, is obtained in Section 1.5.

1.1. Lévy Processes

The reader is referred to Appendices A and B for the basic definitions and facts on Lie groups, stochastic processes, and stochastic analysis.

We will first consider Lévy processes in a general topological group G. A topological group G is a group and a topological space such that both the product map, $G \times G \ni (g, h) \mapsto gh \in G$, and the inverse map, $G \ni g \mapsto g^{-1} \in G$, are continuous. Starting from the next section, we will exclusively consider Lévy processes in Lie groups unless explicitly stated otherwise. A Lie group G is a group and a manifold such that both the product and the inverse maps are smooth. In this work, a manifold is always assumed to be smooth (i.e., C^∞) with a countable base of open sets.

Let G be a topological group and let g_t be a stochastic process in G. For $s < t$, since $g_t = g_s g_s^{-1} g_t = g_t g_s^{-1} g_s$, we will call $g_s^{-1} g_t$ the right increment and $g_t g_s^{-1}$ the left increment of the process g_t over the time interval $(s,\ t)$. The process g_t is said to have independent right (resp. left) increments if these increments over nonoverlapping intervals are independent, that is, if for any $0 < t_1 < t_2 < \cdots < t_n$,

$$g_0,\ g_0^{-1} g_{t_1},\ g_{t_1}^{-1} g_{t_2}, \ldots,\ g_{t_{n-1}}^{-1} g_{t_n}\quad \left(\text{resp. } g_0, g_{t_1} g_0^{-1}, g_{t_2} g_{t_1}^{-1}, \ldots, g_{t_n} g_{t_{n-1}}^{-1}\right)$$

are independent. The process is said to have stationary right (resp. left) increments if $g_s^{-1} g_t \stackrel{d}{=} g_0^{-1} g_{t-s}$ (resp. $g_t g_s^{-1} \stackrel{d}{=} g_{t-s} g_0^{-1}$) for any $s < t$, where $x \stackrel{d}{=} y$ means that the two random variables x and y have the same distribution.

A stochastic process x_t in a topological space is called *càdlàg* (*continu à droite, limites à gauche*) if almost all its paths $t \mapsto g_t$ are right continuous on $\mathbb{R}_+ = [0,\ \infty)$ and have left limits on $(0,\ \infty)$.

A càdlàg process g_t in G is called a left Lévy process if it has independent and stationary right increments. At the moment it may seem more natural to call a left Lévy process a right Lévy process because it is defined using its right increments. However, we call it a left Lévy process because its transition semigroup and generator are invariant under left translations, as will be seen shortly. Similarly, a càdlàg process g_t in G is called a right Lévy process if it has independent and stationary left increments.

Given a filtration $\{\mathcal{F}_t\}$, a left Lévy process g_t in G is called a left Lévy process under $\{\mathcal{F}_t\}$, or a left $\{\mathcal{F}_t\}$-Lévy process, if it is $\{\mathcal{F}_t\}$-adapted and, for any $s < t$, $g_s^{-1} g_t$ is independent of $\mathcal{F}_s$. A right $\{\mathcal{F}_t\}$-Lévy process is defined similarly. Evidently, a left (resp. right) Lévy process is always a left (resp. right) Lévy process under its natural filtration $\{\mathcal{F}_t^0\}$.

If g_t is a left Lévy process, then g_t^{-1} is a right Lévy process, and vice versa. This is a one-to-one correspondence between left and right Lévy processes. There are other ways to establish such a correspondence; for example, if G is a matrix group and g' denotes the matrix transpose of $g \in G$, then $g_t \leftrightarrow g_t'$ gives another one-to-one correspondence between left and right Lévy processes. Because of the duality between the left and right Lévy processes, any result for the left Lévy process has a counterpart for the right Lévy process, and vice versa. We can concentrate only on one of these two processes and derive the results for the other process by a suitable transformation. In the following, we will mainly concentrate on left Lévy processes, except in Chapter 8 and a few other places where it is more natural to work with right Lévy processes.

Let g_t be a left Lévy process in G. Define

$$g_t^e = g_0^{-1} g_t. \tag{1.1}$$

Then g_t^e is a left Lévy process in G starting at the identity element e of G, that is, $g_0^e = e$, and is independent of g_0. Note that, for $t > s$, $(g_s^e)^{-1} g_t^e = g_s^{-1} g_t$.

It is clear that if g_t is a left Lévy process under a filtration $\{\mathcal{F}_t\}$ and if s is a fixed element of $\mathbb{R}_+$, then $g_t' = g_s^{-1} g_{s+t}$ is a left Lévy process identical in distribution to the process g_t^e and independent of $\mathcal{F}_s$. The following proposition says that s may be replaced by a stopping time.

Proposition 1.1. *Let g_t be a left Lévy process under a filtration $\{\mathcal{F}_t\}$. If τ is an $\{\mathcal{F}_t\}$ stopping time with $P(\tau < \infty) > 0$, then under the conditional probability $P(\cdot \mid \tau < \infty)$, the process $g_t' = g_\tau^{-1} g_{\tau+t}$ is a left Lévy process in G that is independent of $\mathcal{F}_\tau$ and has the same distribution as the process g_t^e under P.*

Proof. First assume τ takes only discrete values. Fix $0 < t_1 < t_2 < \cdots < t_k$, $\phi \in C_c(G^k)$ and $\xi \in (\mathcal{F}_\tau)_+$, where $(\mathcal{F}_\tau)_+$ is the set of nonnegative $\mathcal{F}_\tau$-measurable functions. Because $\xi 1_{[\tau=t]} \in (\mathcal{F}_t)_+$, we have

$$\begin{aligned} & E\left[\phi\left(g_\tau^{-1} g_{\tau+t_1}, \ldots, g_\tau^{-1} g_{\tau+t_k}\right)\xi \mid \tau < \infty\right] \\ &= \sum_{t<\infty} E\left[\phi\left(g_t^{-1} g_{t+t_1}, \ldots, g_t^{-1} g_{t+t_k}\right)\xi;\ \tau = t\right] / P(\tau < \infty) \\ &= \sum_{t<\infty} E\left[\phi\left(g_t^{-1} g_{t+t_1}, \ldots, g_t^{-1} g_{t+t_k}\right)\right] E(\xi;\ \tau = t)/P(\tau < \infty) \\ &= E\left[\phi\left(g_0^{-1} g_{t_1}, \ldots, g_0^{-1} g_{t_k}\right)\right] E(\xi \mid \tau < \infty). \end{aligned} \tag{1.2}$$

Setting $\xi = 1$ yields $E[\phi(g_\tau^{-1} g_{\tau+t_1}, \ldots, g_\tau^{-1} g_{\tau+t_k}) \mid \tau < \infty] = E[\phi(g_0^{-1} g_{t_1}, \ldots, g_0^{-1} g_{t_k})]$. Therefore, for a general $\xi \in (\mathcal{F}_\tau^0)_+$, the expression in (1.2) is equal to

$$E[\phi(g_\tau^{-1} g_{\tau+t_1}, \ldots, g_\tau^{-1} g_{\tau+t_k}) \mid \tau < \infty]\, E(\xi \mid \tau < \infty).$$

This proves the desired result for a discrete stopping time τ.

For a general stopping time τ, let $\tau_n = (k+1)2^{-n}$ on the set $[k \cdot 2^{-n} \le \tau < (k+1)2^{-n}]$ for $k = 0, 1, 2, \ldots$. Then τ_n are discrete stopping times and $\tau_n \downarrow \tau$ as $n \uparrow \infty$. The result for τ follows from the discrete case and the right continuity of g_t. □

Let $\mathcal{B}(G)$ be the Borel σ-algebra on G and let $\mathcal{B}(G)_+$ be the space of nonnegative Borel functions on G. For $t \in \mathbb{R}_+$, $g \in G$, and $f \in \mathcal{B}(G)_+$, define

$$P_t f(g) = E\left[f\left(g g_t^e\right)\right]. \tag{1.3}$$

Because g_t is a left Lévy process, for $t > s$ and $f \in \mathcal{B}(G)_+$,

$$E\left[f(g_t) \mid \mathcal{F}_s^0\right] = E\left[f\left(g_s g_s^{-1} g_t\right) \mid \mathcal{F}_s^0\right] = E\left[f\left(h g_{t-s}^e\right)\right]_{h=g_s} = P_{t-s} f(g_s),$$

almost surely. If $g = g_0$, then taking the expectation of this expression, we obtain $P_t f(g) = P_s P_{t-s} f(g)$. This shows that $\{P_t;\, t \in \mathbb{R}_+\}$ is a semigroup of probability kernels on G and that g_t is a Markov process with transition semigroup P_t (see Appendix B.1).

For any $g \in G$, let $L_g\colon G \ni g' \mapsto gg' \in G$ and $R_g\colon G \ni g' \mapsto g'g \in G$ be, respectively, the left and right translations on G. Let $c_g\colon G \ni g' \mapsto gg'g^{-1} \in G$ be the conjugation map on G.

Let H be a subgroup of G. A linear operator T with domain $D(T)$, operating in some function space on G, is called left H-invariant if it is invariant under L_h for all $h \in H$, that is, if

$$\forall h \in H \quad \text{and} \quad f \in D(T), \quad f \circ L_h \in D(T) \quad \text{and} \quad T(f \circ L_h) = (Tf) \circ L_h.$$

Similarly, a right H-invariant operator T is defined using R_h instead of L_h. A left (resp. right) G-invariant operator will simply be called left (resp. right) invariant. A semigroup of probability kernels $\{P_t\}$ will be called left (resp. right) (resp. H-) invariant if each P_t is such an operator on G with domain $\mathcal{B}_b(G)$, the space of all the bounded Borel functions on G. A Markov process with such a transition semigroup will be called left (resp. right) (resp. H-) invariant.

Now assume the topological group G is locally compact and has a countable base of open sets. Then by (1.3) and the right continuity of the process g_t, it can be shown that P_t is a Feller semigroup and is left invariant on G. Therefore, g_t is a Feller process and is left invariant on G. See Appendix B.1 for the definition of Feller processes.

For any measure μ and measurable function f on a measurable space, the integral $\int f\, d\mu$ may be written as $\mu(f)$. In the following, the measurability consideration on a topological space will always refer to the Borel σ-algebra of the space unless explicitly stated otherwise. The convolution of two measures μ and ν on G is a measure $\mu * \nu$ on G defined by

$$\mu * \nu(f) = \int f(gh)\mu(dg)\nu(dh) \tag{1.4}$$

for $f \in \mathcal{B}(G)_+$. A convolution semigroup of probability measures on G is a family $\{\mu_t;\, t \in \mathbb{R}_+\}$ of probability measures on G such that $\mu_0 = \delta_e$ (the unit point mass at e) and $\mu_t * \mu_s = \mu_{t+s}$ for $s, t \in \mathbb{R}_+$. It will be called continuous if $\mu_t \to \delta_e$ weakly as $t \to 0$. Then $\mu_t \to \mu_s$ weakly as $t \downarrow s$ for any $s \in \mathbb{R}_+$.

Let g_t be a left Lévy process in G and let $\{\mu_t;\ t \in \mathbb{R}_+\}$ be the family of the marginal distribution of the process g_t^e; that is, μ_t is the distribution of g_t^e for each $t \in \mathbb{R}_+$. Note that $\mu_t = P_t(e, \cdot)$. Then $\{\mu_t;\ t \in \mathbb{R}_+\}$ is a continuous convolution semigroup of probability measures on G and

$$P_t f(g) = \int f(gh)\mu_t(dh). \tag{1.5}$$

Conversely, let $\{\mu_t;\ t \in \mathbb{R}_+\}$ be a continuous convolution semigroup of probability measures on G. Then P_t defined by (1.5) is a left invariant Feller semigroup. By the discussion in Appendix B.1, there is a càdlàg Markov process g_t in G with transition semigroup P_t and an arbitrary initial distribution. By the Markov property of the process g_t, for $s < t$,

$$E\left[f\left(g_s^{-1}g_t\right)|\mathcal{F}_s^0\right] = P_{t-s}(f \circ L_g)(g_s)\mid_{g=g_s^{-1}} = \mu_{t-s}(f),$$

almost surely, where $\{\mathcal{F}_t^0\}$ is the natural filtration of the process g_t. This shows that the process g_t has independent and stationary right increments; therefore, it is a left Lévy process in G. Note that if g_t is a càdlàg Markov process with a left invariant transition semigroup P_t, then $\mu_t = P_t(e, \cdot)$ is a continuous convolution semigroup of probability measures on G satisfying (1.5), and hence g_t is a left Lévy process.

To summarize, we record the following result:

Proposition 1.2. *Let G be a locally compact topological group with a countable base of open sets.*

(a) A left Lévy process g_t in G is a Markov process with a left invariant Feller transition semigroup P_t given by (1.3). Moreover, the marginal distributions μ_t of the process $g_t^e = g_0^{-1}g_t$ form a continuous convolution semigroup of probability measures on G satisfying (1.5).

(b) If $\{\mu_t;\ t \in \mathbb{R}_+\}$ is a continuous convolution semigroup of probability measures on G and ν is a probability measure on G, then there is a left Lévy process g_t in G with initial distribution ν such that μ_t is the distributions of g_t^e for each $t \in \mathbb{R}_+$.

(c) A left invariant càdlàg Markov process g_t in G is a left Lévy process in G.

1.2. Generators of Lévy Processes

Let M be a manifold. For any integer $k \geq 0$, let $C^k(M)$ be the space of the real- or complex-valued functions on M that have continuous derivatives up to order k with $C(M) = C^0(M)$ being the space of continuous functions on