EXPLORATORY GALOIS THEORY

Combining a concrete perspective with an exploration-based approach, *Exploratory Galois Theory* develops Galois theory at an entirely undergraduate level. The text grounds the presentation in the concept of algebraic numbers with complex approximations and assumes of its readers only a first course in abstract algebra. The author organizes the theory around natural questions about algebraic numbers, and exercises with hints and proof sketches encourage students' participation in the development. For readers with *Maple* or *Mathematica*, the text introduces tools for hands-on experimentation with finite extensions of the rational numbers, enabling a familiarity never before available to students of the subject. *Exploratory Galois Theory* includes classical applications, from ruler-and-compass constructions to solvability by radicals, and also outlines the generalization from subfields of the complex numbers to arbitrary fields. The text is appropriate for traditional lecture courses, for seminars, or for self-paced independent study by undergraduates and graduate students.

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Exploratory Galois Theory

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32 Avenue of the Americas, New York NY 10013-2473, USA

Cambridge University Press is part of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9780521836500

First published 2004

A catalogue record for this publication is available from the British Library

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Library of Congress Cataloguing in Publication data
Swallow, John, 1970–
Exploratory Galois theory / John Swallow.
p. cm.
Includes bibliographical references and index.
ISBN 0-521-83650-6 (hardback) – ISBN 0-521-54499-8 (pbk.)
1. Galois theory. I. Title.
QA214.S93 2004
512′.32 – dc22 2004045196

ISBN 978-0-521-83650-0 Hardback

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Preface

My goal in this text is to develop Galois theory in as accessible a manner as possible for an undergraduate audience.

Consequently, algebraic numbers and their minimal polynomials, objects as concrete as any in field theory, are the central concepts throughout most of the presentation. Moreover, the choices of theorems, their proofs, and (where possible) their order were determined by asking natural questions about algebraic numbers and the field extensions they generate, rather than by asking how Galois theory might be presented with utmost efficiency. Some results are deliberately proved in a less general context than is possible so that readers have ample opportunities to engage the material with exercises. In order that the development of the theory does not rely too much on the mathematical expertise of the reader, hints or proof sketches are provided for a variety of problems.

The text assumes that readers will have followed a first course in abstract algebra, having learned basic results about groups and rings from one of several standard undergraduate texts. Readers do not, however, need to know many results about fields. After some preliminaries in the first chapter, giving readers a common foundation for approaching the subject, the exposition moves slowly and directly toward the Galois theory of finite extensions of the rational numbers. The focus on the early chapters, in particular, is on building intuition about algebraic numbers and algebraic field extensions.

All of us build intuition by experimenting with concrete examples, and the text incorporates, in both examples and exercises, technological tools enabling a sustained exploration of algebraic numbers. These tools assist the exposition in proceeding with a concrete, constructive perspective, and, adopting this point of view, the text presents a Galois theory balanced between theory and computation. The exposition does not, however, fundamentally require or depend on these technological tools, and the text may usefully serve as a balanced introduction to Galois theory even for those who skip the computational sections and exercises.

The particular tools used in the text are contained in `AlgFields`, a package of functions designed for the symbolic computation systems *Maple* and *Mathematica*. This package is freely available for educational use at the website

http://www.davidson.edu/math/swallow/AlgFieldsWeb/index.htm.

The functions are introduced and explained in the occasional sections on computation. These sections treat both *Maple* and *Mathematica* at the same time, since, in general, only minor differences distinguish the syntax of the `AlgFields` package for the two symbolic computation systems. The text uses two-column displays to show input and output, *Maple* on the left and *Mathematica* on the right. (Line breaks are frequently inserted to facilitate the division of the page.) Now just as the text is not a comprehensive treatment of the Galois theory of arbitrary fields, sufficient for preparation for a qualifying exam in algebra in a doctoral program, the routines accompanying the text are not meant to display efficient algorithms for the determination of Galois groups and subfields of field extensions. Instead, the functions provide the ability to ask basic questions about algebraic numbers and to answer these questions using the very same methods and algorithms that appear in the theoretical exposition.

Despite the pedagogical use of computation in the early chapters, by the end of the text, students will be able to place what they have learned from a concrete study of algebraic numbers into a broader context of field theory in characteristic zero. In a pause before the Galois correspondence, the end of the fourth chapter introduces the general concepts of simple, algebraic, and finite extensions and explores the relations among these three properties. After a presentation of the Galois correspondence in the fifth chapter, the text also briefly treats various classical topics in Chapter 6, including cyclic extensions, binomial polynomials, ruler-and-compass constructions, and solvability by radicals.

For those readers for whom this text will be a jumping-off point for a deeper study of Galois theory, the necessary ideas and results for understanding the Galois theory of arbitrary fields are introduced in the penultimate section. That section contains problems leading readers to review previous results in light of a different perspective, one built not on concrete algebraic numbers with complex approximations but on isomorphism classes of arbitrary field extensions. Working through that section, these readers will gain the skills to approach later, with appreciation for the nuances, a more advanced and concise presentation of the subject. Furthermore, even without doing the exercises in that section, readers may profitably apply some of these ideas to finite fields, which are introduced in the final section.

This text may be effectively used in undergraduate major curricula as a second course in abstract algebra, and it would also serve well as a useful guide for a reading course or independent study. The text begins by presenting some standard results on fields in a basic fashion; depending on the content of the reader's first course in abstract algebra, the first chapter and the beginning of the second may be covered quickly. On the other hand, the text ends with a more challenging style, presenting in the last chapter some slightly abbreviated proofs with fewer references to prior theorems. These sections would be suitable for independent work by students in preparation for a class presentation. In fact, the entire text might productively be used for a seminar consisting of a group of students who learn to present this material to their peers; at Davidson College, I have used this material primarily in this fashion.

I would like to thank my many wonderful students at Davidson who have borne the burden of reading various drafts of this text and who have offered so many useful suggestions along the way. These students, Melanie Albert, Sandy Bishop, Frank Chemotti, Brent Dennis, Will Herring, Anders Kaseorg, Margaret Latterner, Chris Lee, Rebecca Montague, Dave Parker, Martha Peed, Joe Rusinko, Andy Schultz, and Ed Tanner, a group who at the time of writing this preface spend their time variously as doctors, graduate students, programmers, teachers, and ultimate players, helped to shape these materials while sharing with me their joy in learning a beautiful subject. I am moreover indebted to Nat Thiem, a coauthor and former summer research student, for his insights as a current graduate student in mathematics.

I also wish to express my great appreciation for friends and colleagues Jorge Aarão, Irl Bivens, Joe Gallian, David Leep, Ján Mináč, Pat Morandi, and Tara Smith, as well as for two reviewers whose names I do not know, for their excellent critiques of various drafts over several years. I am extremely grateful, in particular, to Pierre Dèbes for taking the time to give me such expert advice and judgment on so many topics. I am honored to be part of this community of mathematicians.

It has been a pleasure to work with Roger Astley and his staff at Cambridge University Press, and I thank them for their sound and professional counsel.

Finally, I acknowledge with gratitude the combined support of Davidson College, the Associated Colleges of the South, and the National Science Foundation.

Davidson, North Carolina
March 2004