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Volume 2: Tubes and Concealed Algebras of Euclidean type
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Excerpt
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Chapter X

Tubes

In Chapter VIII of Volume 1, we have started to study the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of any hereditary K-algebra A of Euclidean type, that is, the path algebra $A = KQ$ of an acyclic quiver Q whose underlying graph $\overline{Q}$ is one of the Euclidean diagrams $\widetilde{\mathbb{A}}_m$, with $m \geq 1$, $\widetilde{\mathbb{D}}_m$, with $m \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$. We recall that any such an algebra A is representation-infinite.

We have shown in (VIII.2.3) that the quiver $\Gamma(\operatorname{mod} A)$ contains a unique postprojective component $\mathcal{P}(A)$ containing all the indecomposable projective A-modules, a unique preinjective component $\mathcal{Q}(A)$ containing all the indecomposable injective A-modules, and the family $\mathcal{R}(A)$ of the remaining components being called regular (see (VIII.2.12)). This means that $\Gamma(\operatorname{mod} A)$ has the disjoint union form

$$\Gamma(\operatorname{mod} A) = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A).$$

The indecomposable modules in $\mathcal{R}(A)$ are called regular. We have shown in (VIII.4.5) that there is a similar structure of $\Gamma(\operatorname{mod} B)$, for any concealed algebra B of Euclidean type, that is, the endomorphism algebra

$$B = \operatorname{End} T_A$$

of a postprojective tilting module T_A over a hereditary algebra $A = KQ$ of Euclidean type. The algebra B is representation-infinite.

The objective of Chapters XI-XIII is to describe the structure of regular components of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} B)$ of any concealed algebra B of Euclidean type.

We introduce in this chapter a special type of a translation quiver, which we call a stable tube. The main aim of Section 1 is to describe special properties of irreducible morphisms between indecomposable modules in stable tubes of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} B)$ of an algebra B and their compositions with arbitrary homomorphisms in the module category $\operatorname{mod} B$. In particular, some relevant properties of the radical rad_B and the infinite radical $\operatorname{rad}_B^\infty$ of the category $\operatorname{mod} B$ of finite dimensional right B-modules are described.

In Section 2, we introduce the important concept of a standard component and we prove Ringel's handy criterion on the existence of a standard self-hereditary stable tube in the Auslander–Reiten quiver $\Gamma(\operatorname{mod} B)$ of any algebra B. By applying the criterion, we show in Chapter XI that the regular components of any (representation-infinite) concealed algebra B of Euclidean type are self-hereditary standard stable tubes.

In Section 3, we introduce the concept of a generalised standard component of $\Gamma(\mathrm{mod}\, B)$, invoking the infinite radical rad_B^∞ of the category mod B, and exhibit basic examples of generalised standard components. The main result of Section 4 is a characterisation of (generalised) standard stable tubes obtained by Skowroński in [246], [247], and [254]. It asserts that, for a stable tube $\mathcal{T}$ in the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, B)$ of any algebra B, the following three statements are equivalent:

- $\mathcal{T}$ is a standard stable tube,
- the mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks, and
- $\mathcal{T}$ is a generalised standard stable tube.

It is also shown that $\mathrm{pd}\, X = 1$ and $\mathrm{id}\, X = 1$, for any indecomposable B-module lying in a faithful generalised standard stable tube $\mathcal{T}$ of $\Gamma(\mathrm{mod}\, B)$.

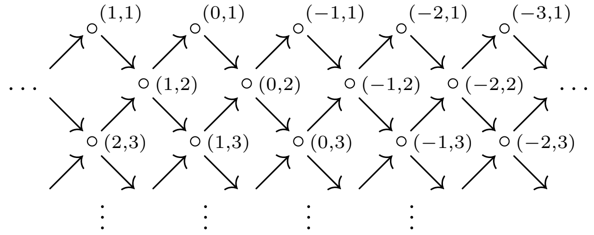
Throughout, we assume that K is an algebraically closed field, and by an algebra we mean a finite dimensional K-algebra. Given a finite quiver $Q = (Q_0, Q_1)$, we denote by KQ the path K-algebra of Q. We recall that the dimension $\dim_K KQ$ of KQ is finite if and only if the quiver Q is **acyclic**, that is, there is no oriented cycle in Q, see Chapters II and III.

X.1. Stable tubes

We have defined in (VIII.1.1) the translation quiver $\mathbb{Z}\Sigma$, for Σ being a connected and acyclic quiver. Thus, letting Σ be the infinite quiver

$$\mathbb{A}_\infty : \quad \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ} \longrightarrow \underset{4}{\circ} \longrightarrow \cdots \longrightarrow \underset{m}{\circ} \longrightarrow \underset{m+1}{\circ} \longrightarrow \cdots$$

we obtain the **infinite translation quiver**

$\mathbb{Z}\mathbb{A}_\infty :$ 

where $\tau(n,i) = (n+1,i)$, for $n \in \mathbb{Z}$ and $i \geq 1$. Thus, by definition, τ is an automorphism of $\mathbb{Z}\mathbb{A}_\infty$, and hence so is any power τ^r of τ (with $r \in \mathbb{Z}$). For a fixed $r \geq 1$, let (τ^r) denote the infinite cyclic group of automorphisms of $\mathbb{Z}\mathbb{A}_\infty$ generated by τ^r, and let $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ denote the orbit space of $\mathbb{Z}\mathbb{A}_\infty$ under the action of (τ^r). That is, $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is the translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each point (n,i) of $\mathbb{Z}\mathbb{A}_\infty$ with the point $\tau^r(n,i) = (n+r,i)$, and each arrow $\alpha : x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r\alpha : \tau^r x \to \tau^r y$. We are thus led to the following definition.

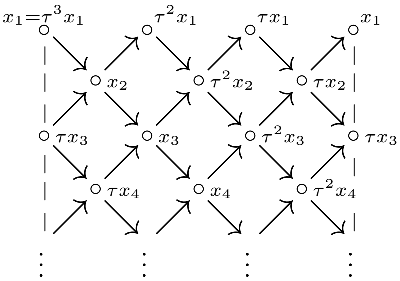
1.1. Definition. Let $(\mathcal{T}, \tau)$ be a translation quiver.

(a) $(\mathcal{T}, \tau)$ is defined to be a **stable tube of rank** $r = r_{\mathcal{T}} \geq 1$ if there is an isomorphism of translation quivers $\mathcal{T} \cong \mathbb{Z}\mathbb{A}_\infty/(\tau^r)$.

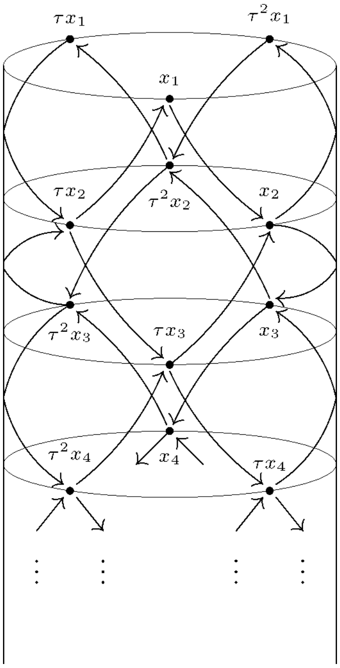
(b) A stable tube of rank $r = 1$ is defined to be a **homogeneous tube**.

(c) Let $(\mathcal{T}, \tau)$ be a stable tube of rank $r \geq 1$. A sequence $(x_1, \dots, x_r)$ of points of $\mathcal{T}$ is said to be a τ**-cycle** if $\tau x_1 = x_r, \tau x_2 = x_1, \dots, \tau x_r = x_{r-1}$.

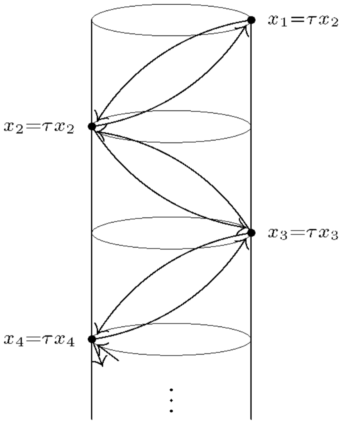
For example, a stable tube of rank 3 is obtained from the quiver



by identifying along the vertical dotted lines, thus giving the following



Similarly, a homogeneous tube has the following shape



We observe that the translation τ still acts as an automorphism over a stable tube of rank r (that is the reason why such tubes are called stable), and that τ^r acts as the identity. The latter fact is expressed by saying that any point of $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is **τ-periodic** of period r.

We recall from Section IX.2 that a path $x_0 \to \cdots \to x_t$ in a translation quiver is called **sectional** if $\tau x_i \ncong x_{i-2}$, for all $i \in \{2, \dots, t\}$.

The following two definitions are of importance in the theory.

1.2. Definition. Let $(\mathcal{T}, \tau)$ be a stable tube.

(a) The set of all points in $\mathcal{T}$ having exactly one immediate predecessor (or, equivalently, exactly one immediate successor) is called the **mouth** of $\mathcal{T}$.

(b) Given a point x lying on the mouth of the stable tube $\mathcal{T}$, a **ray** starting at x is defined to be a unique infinite sectional path

$$x = x[1] \longrightarrow x[2] \longrightarrow x[3] \longrightarrow x[4] \longrightarrow \dots \longrightarrow x[m] \longrightarrow \dots$$

in the tube $\mathcal{T}$.

(c) Given a point x lying on the mouth of the stable tube $\mathcal{T}$, a **coray** ending with x is defined to be a unique infinite sectional path

$$\dots \longrightarrow [m]x \longrightarrow \dots \longrightarrow [4]x \longrightarrow [3]x \longrightarrow [2]x \longrightarrow [1]x = x$$

in the tube $\mathcal{T}$.

To see that the definition is correct, we note that, for each point x lying on the mouth of a stable tube $\mathcal{T}$, there exists a unique arrow starting at x and a unique arrow ending at x. Because an arbitrary point in $\mathcal{T}$ is the source (and the target) of at most two arrows, this implies the existence of a unique infinite sectional path in $\mathcal{T}$ starting at x, and a unique infinite sectional path in $\mathcal{T}$ ending with x.

1.3. Definition. Let A be an algebra and $\mathcal{C}$ be a component of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A.

(a) A **ray point** of $\mathcal{C}$ is defined to be a point X in $\mathcal{C}$ such that there exists an infinite sectional path in $\mathcal{C}$

$$X = X[1] \longrightarrow X[2] \longrightarrow X[3] \longrightarrow X[4] \longrightarrow \dots \longrightarrow X[m] \longrightarrow \dots$$

starting at X and containing all sectional paths starting at X. The corresponding A-module X is called a **ray module**. The unique infinite sectional path starting at X is called the **ray** starting at X.

(b) A **coray point** of $\mathcal{C}$ is defined to be a point X in $\mathcal{C}$ such that there exists an infinite sectional path in $\mathcal{C}$

$$\dots \longrightarrow [m]X \longrightarrow \dots \longrightarrow [4]X \longrightarrow [3]X \longrightarrow [2]X \longrightarrow [1]X = X$$

ending with X and containing all sectional paths ending with X. The corresponding A-module X is called the **coray module**. The unique infinite sectional path ending with X is called the **coray** ending with X.

A **ray point of a stable tube** $\mathcal{T}$ and a **coray point of a stable tube** $\mathcal{T}$ are defined analogously.

It is easy to see that if $(\mathcal{T}, \tau)$ is a stable tube and x is a point x in $\mathcal{T}$, then the following three statements are equivalent:

- x is a ray point of the tube $\mathcal{T}$,
- x is a coray point of $\mathcal{T}$, and
- x lies on the mouth of the tube $\mathcal{T}$.

Now we collect basic facts on the structure of any stable tube of $\Gamma(\operatorname{mod} A)$.

1.4. Lemma. *Let A be an algebra, and $\mathcal{T}$ a stable tube of rank $r = r_{\mathcal{T}} \geq 1$ of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A. Assume that $(X_1, \dots, X_r)$ is a* τ_A**-cycle** *of (indecomposable) mouth modules of mouth A-modules of the tube $\mathcal{T}$, that is, the modules $X_1, \dots, X_r$ lie on the mouth of $\mathcal{T}$ and satisfy $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \dots, \tau_A X_r = X_{r-1}$.*

(a) *For each $i \in \{1, \dots, r\}$, there exists a unique ray*

$$(\mathfrak{r}_i) \quad X_i = X_i[1] \longrightarrow X_i[2] \longrightarrow X_i[3] \longrightarrow \dots \longrightarrow X_i[m] \longrightarrow X_i[m+1] \longrightarrow \dots$$

in $\mathcal{T}$ starting at X_i, and a unique coray

$$(\mathfrak{c}_i) \quad \dots \longrightarrow [m+1]X_i \longrightarrow [m]X_i \longrightarrow \dots \longrightarrow [3]X_i \longrightarrow [2]X_i \longrightarrow [1]X_i = X_i$$

in $\mathcal{T}$ ending with X_i.

(b) *Every indecomposable A-module M in $\mathcal{T}$ is of the form $M \cong X_i[m]$, for some $i \in \{1, \dots, r\}$ and $m \geq 1$.*

(c) *Every indecomposable A-module M in $\mathcal{T}$ is of the form $M \cong [m]X_s$, for some $s \in \{1, \ldots, r\}$ and $m \geq 1$.*

(d) $[m]X_s \cong X_{s-m+1}[m]$ *and* $X_s[m] \cong [m]X_{s+m-1}$ *, for each* $s \in \{1, \ldots, r\}$ *and* $m \geq 1$*, where* $s - m + 1$ *is reduced modulo* $r - 1$ *if* $s - m + 1 \leq 0$ *or* $s - m + 1 \geq r$.

(e) *Under the isomorphisms of A-modules*

$$[1]X_i \cong X_i[1],\ [2]X_i \cong X_{i-1}[2],\ \ldots,\ [m]X_i \cong X_{i+m-1}[m], \ldots$$

the coray $(\mathfrak{c}_i)$ has the form

$$(\mathfrak{c}_i) \quad \ldots \to X_{i-m}[m+1] \longrightarrow X_{i-m+1}[m] \longrightarrow \ldots \longrightarrow X_{i-1}[2] \longrightarrow X_i[1] = X_i.$$

(f) *For any $i \in \{1, \ldots, r\}$ and $m \geq 1$, there exists an almost split sequence*

$$0 \longrightarrow X_i[m] \xrightarrow{\left[\begin{smallmatrix} f_{i,m+1} \\ g_{i,m} \end{smallmatrix}\right]} X_i[m+1] \oplus X_{i+1}[m-1] \xrightarrow{[g_{i,m+1}\ f_{i+1,m}]} X_{i+1}[m] \longrightarrow 0$$

in mod A*, where we set* $X_i[0] = 0$ *and* $X_{i+kr}[j] = X_i[j]$*, for all* $i \in \{1, \ldots, r\}$*,* $j \geq 1$*, and* $k \in \mathbb{Z}$.

Proof. Assume that $(X_1, \ldots, X_r)$ is a τ_A-cycle of mouth modules of the tube $\mathcal{T}$ of rank $r \geq 1$. Then $X_1, \ldots, X_r$ are indecomposable, lie on the mouth of the tube $\mathcal{T}$, and there are isomorphisms $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \ldots, \tau_A X_r = X_{r-1}$. Because the tube $\mathcal{T}$ is stable then there is a surjective morphism $f : \mathbb{Z}\mathbb{A}_\infty \longrightarrow \mathcal{T}$ of translation quivers such that $f(-1,1) = X_1, f(-2,1) = X_2, \ldots, f(-r,1) = X_r$ and the induced morphism $\widetilde{f} : \mathbb{Z}\mathbb{A}_\infty/(\tau^r) \xrightarrow{\simeq} \mathcal{T}$ is an isomorphism. It is clear that the conditions (a), (b), and (c) are satisfied in the translation quiver $(\mathbb{Z}\mathbb{A}_\infty, \tau)$. Hence we easily conclude that (a), (b), and (c) hold in $\mathcal{T}$, if we set $X_i[m] = f(-i,m)$ and $[m]X_i = f(-i+m-1, m)$.

The statement (d) follows from (a), (b), and (c) by an easy induction on $m \geq 1$.

Now we prove (e). It follows from (d) that, given $i \in \{1, \ldots, r\}$ and $m \geq 1$, there are isomorphisms $X_{i-m}[m+1] \cong [m+1]X_i$ and $X_{i-m+1}[m] \cong [m]X_i$. Hence, the following arrow in the coray $(\mathfrak{c}_i)$

$$X_{i-m}[m+1] \cong [m+1]X_i \longrightarrow [m]X_i \cong X_{i-m+1}[m]$$

corresponds to an irreducible morphism $X_{i-m}[m+1] \longrightarrow X_{i-m+1}[m]$ in mod A. To prove (f), we note that in view of the shape of the stable tube $\mathcal{T}$, each of its vertices is a source of at most two arrows, and the arrows correspond to some irreducible morphisms in mod A; thus yield a required almost split sequence. The proof of the lemma is then complete. □

In the remaining part of this section we investigate properties of irreducible morphisms between indecomposable modules in a stable tube $\mathcal{T}$ of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of an algebra A. The investigation needs some new concepts and preliminary results.

Assume that A is an arbitrary algebra. We recall from Section A.3 of Volume 1 that the **radical** $\operatorname{rad}_A = \operatorname{rad}(\operatorname{mod} A)$ of the category $\operatorname{mod} A$ is the two-sided ideal of $\operatorname{mod} A$ defined by the formula

$$\operatorname{rad}_A(X,Y) = \{h \in \operatorname{Hom}_A(X,Y);\ 1_X - gh \text{ is invertible, for any } g : Y \to X\},$$

for each pair of modules X and Y in $\operatorname{mod} A$. If the A-modules X and Y are indecomposable then $\operatorname{rad}_A(X,Y)$ consists of all non-isomorphisms $h : X \longrightarrow Y$ in $\operatorname{mod} A$, see (A.3.4) and (A.3.5) of Volume 1. In particular, $\operatorname{rad}_A(X,X)$ is just the radical of the local algebra $\operatorname{End} X$, for any indecomposable A-module X.

In other words, the radical rad_A of the category $\operatorname{mod} A$ is the two-sided ideal of $\operatorname{mod} A$ generated by all non-isomorphisms between indecomposable A-modules.

Given $m \geq 1$, the mth power $\operatorname{rad}_A^m \subseteq \operatorname{rad}_A$ of rad_A is the two-sided ideal of $\operatorname{mod} A$ such that, for A-modules X and Y, $\operatorname{rad}_A^m(X,Y)$ is the subspace of $\operatorname{rad}_A(X,Y)$ consisting of all finite sums of composite homomorphisms of the form

$$X = X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \longrightarrow \ \ldots \ \longrightarrow X_{m-1} \xrightarrow{h_m} X_m = Y,$$

where $h_j \in \operatorname{rad}_A(X_{j-1}, X_j)$, for any $j \in \{1, 2, \ldots, m-1, m\}$. For $m = 0$, we set $\operatorname{rad}_A^m(X,Y) = \operatorname{Hom}_A(X,Y)$. The intersection

$$\operatorname{rad}_A^\infty = \bigcap_{m=1}^{\infty} \operatorname{rad}_A^m$$

of all powers rad_A^m of rad_A is called the **infinite radical** of $\operatorname{mod} A$.

We recall from (IV.1.6) that a homomorphism $h : X \longrightarrow Y$ between two indecomposable modules X and Y in $\operatorname{mod} A$ is an irreducible morphism if and only if $h \in \operatorname{rad}_A(X,Y) \setminus \operatorname{rad}_A^2(X,Y)$. It follows that the radical rad_A of the category $\operatorname{mod} A$ is the two-sided ideal of $\operatorname{mod} A$ generated by the irreducible morphisms in $\operatorname{mod} A$, as a left ideal and as a right ideal.

Throughout, we need the following two preliminary lemmata.

1.5. Lemma. *Let A be an algebra, and M, N be a pair of modules in $\operatorname{mod} A$.*

(a) *There exists an integer $m \geq 0$ such that $\operatorname{rad}_A^\infty(M,N) = \operatorname{rad}_A^m(M,N)$.*
(b) *$\operatorname{rad}_A^\infty(M,N) = \operatorname{rad}_A(M,N)$, if the modules M and N are indecomposable and lie in two different components $\mathcal{C}_1$ and $\mathcal{C}_2$ of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A.*

Proof. (a) Because the vector space $\operatorname{Hom}_A(M,N)$ is finite dimensional, the descending chain

$$\mathrm{Hom}_A(M,N) \supseteq \mathrm{rad}_A(M,N) \supseteq \mathrm{rad}_A^2(M,N) \supseteq \ldots \supseteq \mathrm{rad}_A^m(M,N) \supseteq \ldots$$

terminates, that is, $\mathrm{rad}_A^m(M,N) = \mathrm{rad}_A^{m+1}(M,N) = \mathrm{rad}_A^{m+2}(M,N) = \ldots,$ for some $m \geq 0$. It follows that $\mathrm{rad}_A^\infty(M,N) = \mathrm{rad}_A^m(M,N)$.

The statement (b) is a consequence of (IV.5.1). □

1.6. Lemma. *Let A be an algebra, M an A-module, and*

$$0 \longrightarrow Z \xrightarrow{\left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right]} X \oplus X' \xrightarrow{[f,\ f']} Y \longrightarrow 0$$

an almost split sequence in $\mathrm{mod}\, A$*, where X, X', Y, and Z are indecomposable modules.*

(a) *Let $\ell \geq 1$ be an integer and $h : M \longrightarrow X$ be a homomorphism in* $\mathrm{mod}\, A$ *such that $h \notin \mathrm{rad}_A^\ell(M,X)$ and $fh \in \mathrm{rad}_A^{\ell+1}(M,Y)$. Then there exists a homomorphism $h' : M \longrightarrow Z$ in* $\mathrm{mod}\, A$ *such that $h' \notin \mathrm{rad}_A^{\ell-1}(M,Z)$ and $g'h' \in \mathrm{rad}_A^\ell(M,X')$.*

(b) *Let $\ell \geq 1$ be an integer and $t : X \longrightarrow M$ be a homomorphism in* $\mathrm{mod}\, A$ *such that $t \notin \mathrm{rad}_A^\ell(X,M)$ and $tg \in \mathrm{rad}_A^{\ell+1}(Z,M)$. Then there exists a homomorphism $t' : Y \longrightarrow M$ in* $\mathrm{mod}\, A$ *such that $t' \notin \mathrm{rad}_A^{\ell-1}(Y,M)$ and $t'f' \in \mathrm{rad}_A^\ell(X',M)$.*

Proof. We only prove the statement (a), because the proof of (b) is similar.

Assume that $h \notin \mathrm{rad}_A^\ell(M,X)$ and $fh \in \mathrm{rad}_A^{\ell+1}(M,Y)$. Then there exists an A-module N and two homomorphisms $w \in \mathrm{rad}_A^\ell(M,N)$ and $v \in \mathrm{rad}_A(N,Y)$ such that $fh = vw$. Then v is not a retraction and, hence, there exists a homomorphism $\left[\begin{smallmatrix} u \\ u' \end{smallmatrix}\right] : N \longrightarrow X \oplus X'$ such that $v = [f,\ f'] \cdot \left[\begin{smallmatrix} u \\ u' \end{smallmatrix}\right] = fu + f'u'$. Consider the homomorphisms

$$\left[\begin{smallmatrix} uw-h \\ u'w \end{smallmatrix}\right] : M \longrightarrow X \oplus X'$$

and note that

$$[f,\ f'] \cdot \left[\begin{smallmatrix} uw-h \\ u'w \end{smallmatrix}\right] = fuw - fh + f'u'w = (fu + f'u')w - fh = vw - fh = 0.$$

It follows that there exists a homomorphism $h' : N \longrightarrow Z$ such that

$$\left[\begin{smallmatrix} uw-h \\ u'w \end{smallmatrix}\right] = \left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right] \cdot h',$$

and therefore $g'h' = u'w \in \mathrm{rad}_A^\ell(M,X')$, because $\left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right]$ is the kernel of $[f,\ f']$. Note also that $h' \notin \mathrm{rad}_A^{\ell-1}(M,Z)$, because otherwise we get the contradiction $h = uw - gh' \in \mathrm{rad}_A^\ell(M,X)$. This finishes the proof. □

1.7. Proposition. *Let A be an algebra, $\mathcal{T}$ a stable tube of rank $r = r_\mathcal{T} \geq 1$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A, and $(X_1, \ldots, X_r)$ a τ_A-cycle of mouth modules of the tube $\mathcal{T}$. In the notation of* (1.4)*, let* $(\mathfrak{r}_i)$

and $(\mathfrak{c}_i)$ *be the ray starting at* X_i *and the coray ending with* X_i *in* $\mathcal{T}$, *for each* $i \in \{1, \dots, r\}$.

(a) *For each* $i \in \{1, \dots, r\}$ *and* $m \geq 2$, *any irreducible morphism* $f_{i,m} : X_i[m-1] \longrightarrow X_i[m]$ *corresponding to the arrow* $X_i[m-1] \longrightarrow X_i[m]$ *in the ray* $(\mathfrak{r}_i)$ *of* $\mathcal{T}$ *starting at* X_i *is a monomorphism.*

(b) *For each* $i \in \{1, \dots, r\}$ *and* $m \geq 2$, *any irreducible morphism* $g_{i,m} : [m]X_i \longrightarrow [m-1]X_i$ *corresponding to the arrow* $[m]X_i \longrightarrow [m-1]X_i$ *in the coray* $(\mathfrak{c}_i)$ *of* $\mathcal{T}$ *ending with* X_i *is an epimorphism.*

(c) *For each* $i \in \{1, \dots, r\}$ *and* $m \geq 2$, *there exist irreducible morphisms*

$$v_{i,m} : X_i[m-1] \longrightarrow X_i[m] \quad \text{and} \quad q_{i,m} : X_i[m] \longrightarrow X_{i+1}[m-1]$$

in mod A *such that*

(c1) $q_{i,2}v_{i,2} \in \operatorname{rad}_A^3(X_i[1], X_{i+1}[1])$, *and*

(c2) $v_{i+1,m}q_{i,m} + q_{i,m+1}v_{i,m+1} \in \operatorname{rad}_A^3(X_i[m], X_{i+1}[m])$.

Proof. Assume that $\mathcal{T}$ is a stable tube of rank $r = r_{\mathcal{T}} \geq 1$ of $\Gamma(\operatorname{mod} A)$ and $(X_1, \dots, X_r)$ is a τ_A-cycle of mouth modules of the tube $\mathcal{T}$, that is, the modules $X_1, \dots, X_r$ are indecomposable, lie on the mouth of the tube $\mathcal{T}$, and there are isomorphisms $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \dots, \tau_A X_r = X_{r-1}$. Then (1.4) applies, and we freely use the notation introduced there. Clearly, for each $i \in \{1, \dots, r\}$ and $m \geq 2$, the arrow $X_i[m-1] \longrightarrow X_i[m]$ in the ray $(\mathfrak{r}_i)$ of $\mathcal{T}$ starting from X_i corresponds to an irreducible morphism $f_{i,m} : X_i[m-1] \longrightarrow X_i[m]$ in mod A. By (1.4)(d), there is an isomorphism $[m]X_s \cong X_{s-m+1}[m]$, for each $s \in \{1, \dots, r\}$ and $m \geq 1$, where $s-m+1$ is reduced modulo $r-1$ if $s-m+1 \leq 0$ or $s-m+1 \geq r$. It follows that, given $i \in \{1, \dots, r\}$ and $m \geq 1$, there are isomorphisms $X_i[m] \cong [m]X_{i+m-1}$ and $X_{i+1}[m-1] \cong [m-1]X_{i+m-1}$. Hence, the arrow

$$X_i[m] \cong [m]X_{i+m-1} \longrightarrow [m-1]X_{i+m-1} \cong X_{i+1}[m-1]$$

in the coray $(\mathfrak{c}_i)$ of $\mathcal{T}$ ending with X_i corresponds to an irreducible morphism $g_{i,m} : X_i[m] \longrightarrow X_{i+1}[m-1]$ in mod A. In view of the shape of the stable tube $\mathcal{T}$, each of its vertices is a source of at most two arrows. It follows that, for any $i \in \{1, \dots, r\}$ and $m \geq 1$, there exists an almost split sequence

$$0 \longrightarrow X_i[m] \xrightarrow{\begin{bmatrix} f_{i,m+1} \\ g_{i,m} \end{bmatrix}} X_i[m+1] \oplus X_{i+1}[m-1] \xrightarrow{[g_{i,m+1} \ f_{i+1,m}]} X_{i+1}[m] \longrightarrow 0$$

in mod A, where we set $X_i[0] = 0$ and $X_{i+kr}[j] = X_i[j]$, for all $i \in \{1, \dots, r\}$, $j \geq 1$, and $k \in \mathbb{Z}$. Counting the dimensions we get

$$\dim_K X_i[m] + \dim_K X_{i+1}[m] = \dim_K X_i[m+1] + \dim_K X_i[m-1].$$

Hence, we conclude the inequalities $\dim_K X_i[1] < \dim_K X_i[2] > \dim_K X_{i+1}[1]$, and an easy induction on $m \geq 1$ shows that

$$\dim_K X_i[m] < \dim_K X_i[m{+}1] > \dim_K X_{i+1}[m],$$

for all $i \in \{1, \dots, r\}$ and $m \geq 1$. Hence we easily conclude that any irreducible morphism $f_{i,m} : X_i[m{-}1] \longrightarrow X_i[m]$ is a monomorphism and any irreducible morphism $g_{i,m} : [m]X_i \longrightarrow [m{-}1]E_i$ is an epimorphism, because we know from (IV.1.4) that every irreducible morphism in mod A is a proper monomorphism or a proper epimorphism. This finishes the proof of (a) and (b).

Now we prove (c). First we observe that if $f, g : X \longrightarrow Y$ are irreducible morphisms in mod A and X, Y are indecomposable modules lying in the tube $\mathcal{T}$ then

$$f + \mathrm{rad}_A^2(X, Y) = \lambda \cdot g + \mathrm{rad}_A^2(X, Y),$$

for some $\lambda \in K \setminus \{0\}$, because

$$\dim_K \mathrm{rad}_A(X, Y)/\mathrm{rad}_A^2(X, Y) = 1,$$

by (IV.1.6), (IV.4.6), and the definition of a stable tube. For each $i \in \{1, \dots, r\}$, we choose some irreducible morphisms

$$v_{i,2} : X_i[1] \longrightarrow X_i[2] \quad \text{and} \quad q_{i,2} : X_i[2] \longrightarrow X_{i+1}[1]$$

in mod A, and consider an almost split sequence

$$0 \longrightarrow X_i[1] \xrightarrow{u_{i,2}} X_i[2] \xrightarrow{p_{i,2}} X_{i+1}[1] \longrightarrow 0,$$

in mod A. By our earlier observations, there exist scalars $\lambda_i, \mu_i \in K \setminus \{0\}$ and homomorphisms $w_{i,2} \in \mathrm{rad}_A^2(X_i[1], X_i[2])$ and $t_{i,2} \in \mathrm{rad}_A^2(X_i[2], X_{i+1}[1])$ such that $v_{i,2} = \lambda_i u_{i,2} + w_{i,2}$ and $q_{i,2} = \mu_i p_{i,2} + t_{i,2}$. Then we get the equalities

$$\begin{aligned} q_{i,2}v_{i,2} &= (\mu_i p_{i,2} + t_{i,2})(\lambda_i u_{i,2} + w_{i,2}) \\ &= \mu_i \lambda_i p_{i,2} u_{i,2} + (\mu_i p_{i,2} w_{i,2} + \lambda_i t_{i,2} u_{i,2} + t_{i,2} w_{i,2}) \\ &= \mu_i p_{i,2} w_{i,2} + \lambda_i t_{i,2} u_{i,2} + t_{i,2} w_{i,2} \in \mathrm{rad}_A^3(X_i[1], X_{i+1}[1]), \end{aligned}$$

and (c1) follows.

Assume that $s \geq 2$ and, for all $i \in \{1, \dots, r\}$ and $k \in \{2, \dots, s\}$, there exist irreducible morphisms

$$v_{i,k} : X_i[k{-}1] \longrightarrow X_i[k] \quad \text{and} \quad q_{i,k} : X_i[k] \longrightarrow X_{i+1}[k{-}1]$$

in mod A satisfying (c1) and (c2), for $k \in \{2, \dots, s-1\}$. Fix $i \in \{1, \dots, r\}$ and consider an almost split sequence

$$0 \longrightarrow X_i[s] \xrightarrow{\left[\begin{smallmatrix} u_{i,s+1} \\ p_{i,s} \end{smallmatrix}\right]} X_i[s{+}1] \oplus X_{i+1}[s{-}1] \xrightarrow{[p_{i,s+1}\ u_{i+1,s}]} X_{i+1}[s] \longrightarrow 0$$

in mod A. Then there exist scalars $\lambda_{i+1}^{(s)}, \mu_i^{(s)} \in K \setminus \{0\}$ and homomorphisms $w_{i+1,s} \in \mathrm{rad}_A^2(X_{i+1}[s{-}1], X_{i+1}[s])$ and $t_{i,s} \in \mathrm{rad}_A^2(X_i[s], X_{i+1}[s{-}1])$ such that $v_{i+1,s} = \lambda_{i+1}^{(s)} u_{i+1,s} + w_{i+1,s}$ and $q_{i,s} = \mu_i^{(s)} p_{i,s} + t_{i,s}$.