

Economic Growth and Macroeconomic Dynamics

Recent Developments in Economic Theory

Edited by

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1

Growth and the Elasticity of Factor Substitution

John D. Pitchford

One measure of the shape of production isoquants is the elasticity of substitution between factors. It ranges in value from zero to infinity, implying that no substitution is possible when it is zero and that factors are perfect substitutes when it is infinity. It has been a limitation on the generality of the conclusions of growth models that explicit treatment of substitution has largely been confined to cases in which the elasticity of substitution between labor and capital is unity. This limitation is imposed by the use of the Cobb–Douglas production function.¹ This chapter is based on Professor Swan’s growth model, but the Cobb–Douglas production function is replaced by a production function which allows the elasticity of substitution to take any value between zero and infinity. It is seen that a variety of growth paths is possible, depending on the elasticity of substitution, and this leads to a reconsideration of the relation between income growth and the saving ratio.

¹ Solow, *op. cit.*, does consider the case in which the elasticity of substitution is 2. T. W. Swan’s model, “Economic Growth and Capital Accumulation,” *Economic Record*, 1956, uses the Cobb–Douglas function.

The development of this article has benefited from discussions with B. Thalberg and T. N. Srinivasan at Yale, and K. Frearson of the University of Melbourne. I am also indebted to Professor T. W. Swan and Dr. I. F. Pearce of the Australian National University and Professor R. M. Solow of the Massachusetts Institute of Technology, who made useful comments on an earlier draft. The production function I have used was employed by R. M. Solow in a talk at Yale titled “Substitution between Capital and Labour.” Professor Solow discussed this function in connection with procedures for estimating the elasticity of substitution. A similar function appears in his article, “A Contribution to the Theory of Economic Growth,” *Quarterly Journal of Economics*, 1956, p. 77.

I

Because the model differs from Swan's only in the substitution possibilities which it allows I shall not explain in detail the meaning of the system.²

Symbols

$$Y\text{-income; } y = \frac{dY}{dt} \cdot \frac{1}{Y};$$

$$K\text{-capital; } k = \frac{dK}{dt} \cdot \frac{1}{K};$$

$$N\text{-labor; } n = \frac{dN}{dt} \cdot \frac{1}{N};$$

σ —the elasticity of substitution between capital and labor

s —the average *equals* the marginal saving ratio.

Savings are assumed equal to investment and the marginal product of labor equal to the real wage throughout.

The first assumption gives

$$\frac{dK}{dt} \cdot \frac{1}{K} = k = sY/K. \quad (1)$$

The second ensures that labor offering for employment is always equal to the demand for labor.

The production function is

$$Y = [\gamma K^{-\beta} + \mu N^{-\beta}]^{-\frac{1}{\beta}}, \quad (2)$$

where $\beta = (1 - \sigma)/\sigma$,³ $\gamma = j(\beta)$, and $\mu = h(\beta)$ so that when $\beta = 0$, $\gamma + \mu = 1$. It is necessary to impose this restriction on the values of γ and μ when $\beta = 0$ (i.e., $\sigma = 1$) in order to ensure that for all values of β the function exhibits constant returns to scale. This is ensured for values of β other than zero by raising $(\gamma K^{-\beta} + \mu N^{-\beta})$ to the power $-1/\beta$.

This function then has the elasticity of substitution as a parameter, for σ may be given any value from zero to infinity by letting β take an appropriate value in the range of infinity to minus unity.

² The limitations which his simplifying assumptions produce apply also to my model.

³ Thus, when $0 < \sigma < 1$, $\infty > \beta > 0$; and when $0 < \sigma < \infty$, $0 > \beta > -1$.

For any differentiable function $Y = f(K, N)$, where Y , N , and K are functions of t , we may write

$$\frac{dY}{dt} = \frac{\partial Y}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial Y}{\partial N} \cdot \frac{dN}{dt},$$

and, hence,

$$\frac{dY}{dt} \frac{1}{Y} = \frac{\partial Y}{\partial K} \frac{K}{Y} \cdot \frac{dK}{dt} \frac{1}{K} + \frac{\partial Y}{\partial N} \frac{N}{Y} \cdot \frac{dN}{dt} \frac{1}{N}$$

or $y = \epsilon_K k + \epsilon_N n$, where ϵ_K and ϵ_N are the production elasticities of capital and labor, respectively.

From (2) we have

$$\epsilon_K = \gamma \left(\frac{Y}{K} \right)^\beta,$$

and

$$\epsilon_N = \mu \left(\frac{Y}{N} \right)^\beta.$$

Thus, in terms of the rates of growth of product and factors, (2) may be written

$$y = \gamma \left(\frac{Y}{K} \right)^\beta k + \mu \left(\frac{Y}{N} \right)^\beta n. \quad (3)$$

Because we are assuming constant returns to scale, we must also have

$$y = \gamma \left(\frac{Y}{K} \right)^\beta k + \left[1 - \gamma \left(\frac{Y}{K} \right)^\beta \right] n. \quad (4)$$

Swan's model is depicted on a diagram with growth rates on the vertical and the output-capital ratio on the horizontal axis. On this diagram the labor force growth rate (assumed constant) appears as a horizontal straight line, while the capital growth rate ($k = s(Y/K)$) is a straight line through the origin with slope s . The output growth line completes the system. In the Swan model it is given by $y = \epsilon_K k + (1 - \epsilon_K) n$, where ϵ_K and $1 - \epsilon_K$ are the *constant* production elasticities attached to capital and labor, respectively.

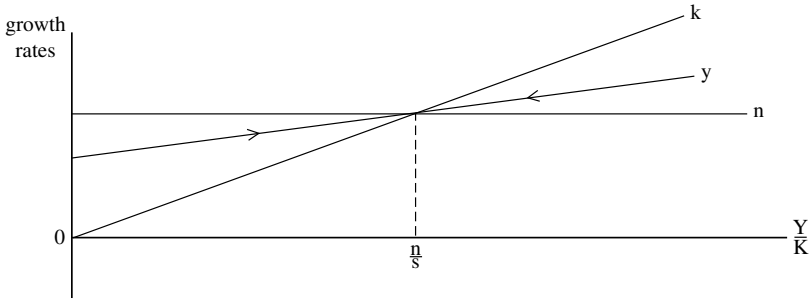


Figure 1. Swan Diagram.

It follows from (3) that when $\sigma = 1$ ($\beta = 0$), Swan's solution emerges as a special case for

$$y = \gamma \left(\frac{Y}{K} \right)^0 k + \mu \left(\frac{Y}{N} \right)^0 n$$

$$\therefore y = \gamma k + \mu n.$$

This system is shown in Figure 1. A stable (*golden age*) equilibrium is seen to exist when $y = k = n$, and $Y/K = n/s$. This equilibrium will involve the same rate of growth of income whatever the saving ratio. Moreover, as (during the process of adjustment from one equilibrium to another) “‘plausible’ figuring suggests that even the impact effect of a sharp rise in the saving ratio may be of minor importance for the rate of growth”⁴ saving is seen to be unimportant as an influence on the income growth rate.

We should not, however, be misled into ignoring the effect which an increase in the saving ratio will have on the *level*, as distinct from the equilibrium *rate of growth*, of income. A rise in the saving ratio increases output per head and, hence, raises the base upon which income grows.⁵

II

Let us now allow for the full range of possible values of the elasticity of substitution by employing the production function given by Equation

⁴ Swan, *op. cit.*, p. 338.

⁵ *Ibid.* For the Cobb–Douglas production function it may be shown that $Y/K = (Y/N)^{\epsilon_K/\epsilon_N}$, from which the preceding results follow.

(2).⁶ This function is found to operate only for a limited range of the values of Y/K . Rearranging (2), we have

$$\frac{Y}{K} = \left[\gamma + \mu \left(\frac{K}{N} \right)^\beta \right]^{-\frac{1}{\beta}}.$$

Now when the elasticity of substitution is greater than unity ($\beta < 0$) the output–capital ratio is seen to have a lower limit, $(1/\gamma)^{\frac{1}{\beta}}$, because any value of Y/K below this would require a capital–labor ratio greater than infinity. Thus at the limiting value of Y/K the capital–labor ratio would have to be infinite. When the elasticity of substitution is less than unity ($\beta > 0$) there is an upper limit to Y/K of $(\frac{1}{\gamma})^{\frac{1}{\beta}}$, and at this upper limit it can be seen that the capital–labor ratio will be zero.

These limiting values of the output–capital ratio are shown in the following diagrams. Figures 2(a), 2(b), 2(c), and 2(d) illustrate the growth paths which the model may take; the shape of the income growth line being based on propositions which are obtained in the Appendix.

If the elasticity of substitution is less than unity, Figures 2(a) and 2(b) will be relevant, whilst Figures 2(c) and 2(d) apply to cases in which the elasticity of substitution is greater than unity. If $\sigma < 1$, Figure 2(a) is more likely than Figure 2(b), the higher the output–capital ratio appropriate to a golden age (n/s) , and the lower the limiting value of the output–capital ratio $[(1/\gamma)^{\frac{1}{\beta}}]$. $(Y/K) = n/s$ will be higher the greater the population growth rate and the lower the saving ratio. If the population growth rate is higher than the saving ratio $(n/s > 1)$, $(1/\gamma)^{\frac{1}{\beta}}$ must also be greater than unity in order for Figure 2(b) to be applicable. β in this case is positive, so that in order for $(1/\gamma)^{\frac{1}{\beta}}$ to be greater than unity γ must be smaller than unity. On the other hand, if $n/s < 1$, a value of γ smaller than unity will not be necessary to make Figure 2(b) relevant.

If $\sigma > 1$, Figure 2(d) is more likely than Figure 2(c) the lower n/s , and the higher $(1/\gamma)^{\frac{1}{\beta}}$. Thus, the greater the saving ratio and the smaller the rate of population growth the more probable will be

⁶ The case in which $\sigma = 0, \beta = \infty$ is not explicitly treated in what follows. When there is no substitution between factors we have the elements of the simplified Harrod and Joan Robinson models. There is an excellent treatment of the Harrod case in the literature (Solow, *op. cit.*). When $\sigma = \infty, \beta = -1$, the production function reduces to $Y = \gamma K + \mu N$, which may be rewritten $y = \gamma (K/Y)k + \mu (N/Y)n$, and yields the same sorts of results as the more general form.

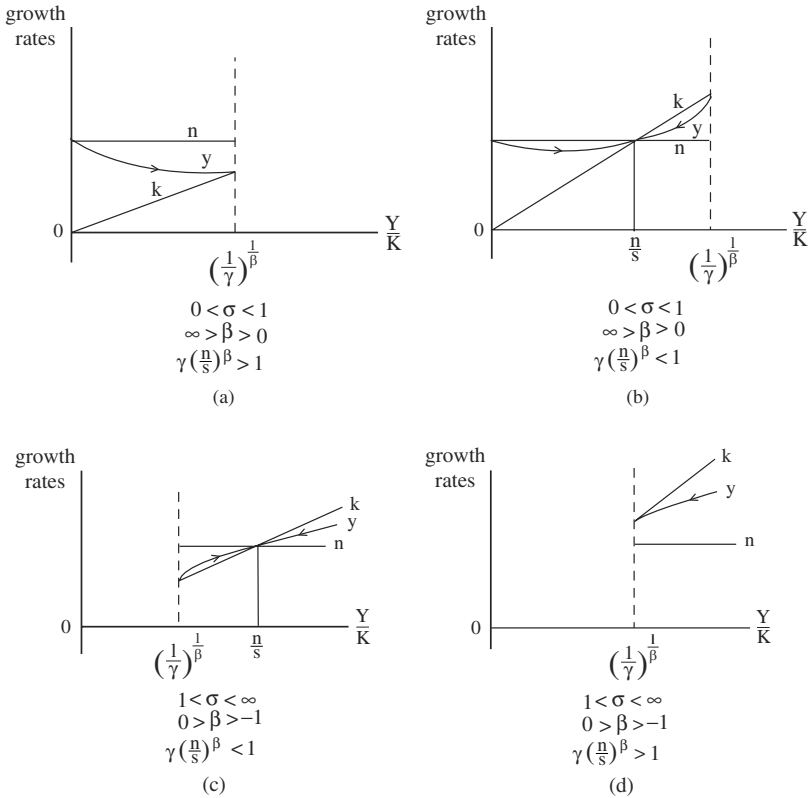


Figure 2. Growth and the Elasticity of Substitution.

Figure 2(d). If $n/s > 1$, $(1/\gamma)^{\frac{1}{\beta}}$ must be greater than unity. This would require (with $\beta < 0$) a value of γ greater than unity. With $n/s < 1$, γ need not be greater than unity in order to ensure that Figure 2(d) applies.

Our knowledge of the values of some of these parameters does not help us to make a choice between these four cases. We know that usually $n < s$, but we do not know anything about the value of γ , nor about the value of σ (unless we take the fitting of Cobb–Douglas production functions to suggest that it is in the neighborhood of unity). Even if we did know the value of σ it would still be necessary to know γ before we could choose between Figures 2(a) and 2(b) ($\sigma < 1$) and between Figures 2(c) and 2(d) ($\sigma > 1$).

Before we examine each of these behavior paths it is useful to look at some of the elementary propositions about growth with constant returns to scale.⁷

We have that

$$y = (1 - \epsilon_N)k + \epsilon_N n$$

for a production function subject to constant returns to scale;

$$\begin{aligned} \therefore y - k &= \epsilon_N n - \epsilon_N k \\ &= \epsilon_N s \left(\frac{n}{s} - \frac{Y}{K} \right). \end{aligned} \quad (5)$$

Given $\epsilon_N > 0$, this system is seen to be stable and to approach a (golden age) equilibrium in which $Y/K = n/s$ or $y = k = n$.

Thus, provided $\epsilon_N > 0$, a golden age is always approached when there are constant returns to scale and no technical progress.

This suggests the basis of distinction between the two different types of behavior which the model can produce. In those cases in which the system grows towards a golden age (Figures 2(b) and 2(c)) the labor production elasticity (ϵ_N) must be positive throughout the process, whereas if a golden age is not approached forces must be in operation to push the labor elasticity to (a limiting position of) zero.

The two golden age cases do not require much explanation, for the shifts in the production elasticities and the marginal productivities which bring the system to equilibrium may be inferred from a consideration of the diagrams and Equations (3) and (4). These cases, of course, obey the rule that the income growth rate is, in equilibrium, uninfluenced by the saving ratio. It is the two cases in which a golden age is not possible that invite detailed examination.

As we have seen in both these cases the contribution of labor to the productive process eventually becomes negligible in the sense that, after a point, further increases in the labor force employed fail to increase output significantly. Figure 2(a) ($\sigma < 1$) involves the labor force growing more rapidly than the capital stock. Because the capital-labor ratio is continually falling, labor must be increasingly substituted for capital in order to maintain full employment of both factors. The fact that labor and capital are poor substitutes will mean that more and

⁷ See T. W. Swan, "Golden Ages and Production Functions," in K. E. Berrill (ed.), *Economic Development with Special Reference to East Asia*, London; MacMillan, 1963.

more labor can be employed only if the real wage (*equals* the marginal product of labor) is forced down. In this case the marginal product of labor falls more rapidly than output per head (i.e., $\partial Y/\partial N$ falls more rapidly than N/Y rises) so that the labor production elasticity, $(\partial Y/\partial N) \cdot (N/Y) = \mu(Y/N)^\beta$, declines. Before a golden age can be reached the capital–labour ratio has tended to zero so that the fall in the capital–output ratio comes to a halt. It follows that if, in equilibrium, the labor elasticity is zero, output (with constant returns to scale) must grow at the same rate as capital. Hence, as capital grows more slowly than labor, output must grow at a less than golden age rate. Of course this equilibrium will be reached only after infinite time has elapsed, but it can be stated that in the circumstances in which Figure 2(a) holds income will grow towards such an equilibrium, and during this process the income growth rate will always be less than the labor growth rate.

The decline in the marginal product of labor implies a fall in the real wage. Before real wages fall to zero the labor force growth rate will decline (either because population growth is reduced by a Malthusian process, or because unemployment develops). As long as some accumulation is taking place the result will be eventually to render a golden age possible (i.e., to ensure $\gamma(n/s)^\beta < 1$). However, as the labor growth rate has fallen the income growth rate will be less than the initial labor growth rate and there may be some unemployed labor at the new equilibrium.

Figure 2(d) ($\sigma > 1$) involves income growing permanently at a higher rate than labor. This causes a rise in the capital–output ratio; for when income grows faster than employment, with constant returns to scale, capital must be growing more rapidly than income. This deepening of capital would eventually produce a golden age, except that in this case the capital–labour ratio becomes infinite before such an equilibrium can be reached. Capital and labor are good substitutes in this situation, and capital is increasingly substituted for labor as the process proceeds. The marginal product of labor is raised by this substitution, but, nevertheless, as in the previous case, the labor elasticity (ϵ_N) tends to zero as the limiting value of the output–capital ratio is approached.

This case is associated with a high saving ratio and/or a low population growth and a high value of the constant attached to capital (γ). The labor production elasticity must eventually fall to zero because the community eventually has such a large stock of capital compared

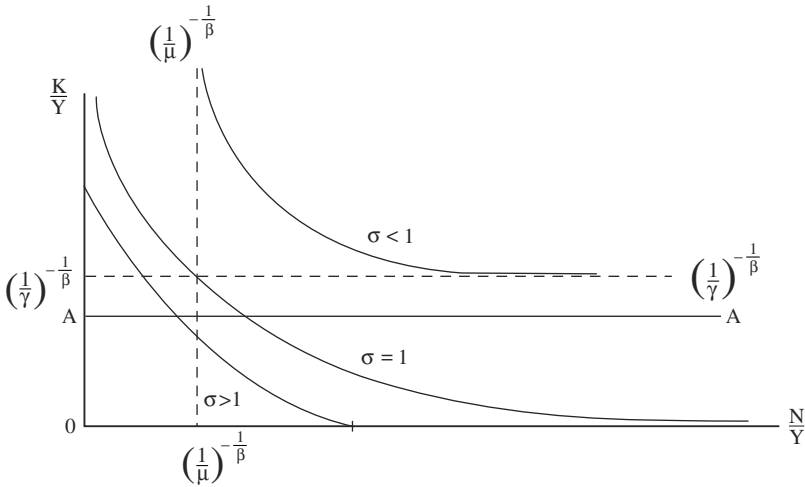


Figure 3. Isoquants.

to the stock of labor, and this capital is a good substitute for labor, so that a given percentage change in the labor force produces a negligible percentage change in the level of output.

All this can be looked at in terms of the shape and position of the production isoquants for different values of the elasticity of substitution. The production Function (2) may be stated in the form of a relationship between K/Y and N/Y . Thus,

$$\frac{K}{Y} = \left[\frac{1}{\gamma} - \frac{\mu}{\gamma} \left(\frac{N}{Y} \right)^{-\beta} \right]^{-\frac{1}{\beta}} \tag{6}$$

In the Appendix it is shown that this relationship will involve the forms shown in Figure 3 for different values of σ .

For any given level of income these curves illustrate the possible shapes of the production isoquants. When $\sigma > 1$, (6) will be asymptotic to positive limits with respect to both K/Y and N/Y . When $\sigma = 1$, (6) will be asymptotic to both axes, whilst when $\sigma < 1$, (6) will cut both axes at finite values.

Now in a golden age $k = s(Y/K) = n$, so that for an economy to attain a golden age it must attain a capital-output ratio such that $K/Y = s/n$. The line AA in Figure 3 is one such equilibrium value of K/Y . It is clear that the Cobb-Douglas production function can attain any

capital–output ratio, so that, from any initial value, growth will take place along the curve for $\sigma = 1$ until the appropriate value of K/Y is reached. On the other hand, if $\sigma \neq 1$, it can be seen that only if the line AA cuts Equation (6) will a golden age be possible. In Figure 3, AA is drawn so that with the curve given for $\sigma > 1$ a golden age can be reached, but in the case of the curve for $\sigma < 1$ a movement downward and to the right can never attain the required value of s/n .

III

One interesting implication of these processes is that in some circumstances a rise in the saving ratio can achieve a permanently higher rate of growth of income. Swan had concluded that “[A]fter a transitional phase, the influence of the saving ratio on the rate of growth is ultimately absorbed by a compensating change in the output–capital ratio.”⁸ However, he had not examined the possibility of the labor elasticity becoming zero and, thus, had not allowed for cases such as Figures 2(a) and 2(d).

Only when substitution is difficult and a golden age is achievable will it be impossible permanently to raise the rate of growth of income by raising the saving ratio.⁹ As we have seen, when substitution is difficult and a golden age is not possible, income will always grow at a rate less than the golden age growth rate. An appropriate rise in the saving ratio (provided this can be achieved) will make a golden age possible so that income can eventually grow at the same rate as labor.

In a golden age, when substitution is easy, a higher income growth rate may be produced by raising the saving ratio. This means that the higher saving changes the process from the type shown by Figure 2(c) to the type shown by Figure 2(d).

Apart from the possibility of shifting from one diagram to another it is possible in Figures 2(a) and 2(d) to raise the *equilibrium* growth rate by raising the saving ratio. Raising s does not influence the value of $(\frac{1}{\gamma})^{\frac{1}{\beta}}$ (the limit to the values of Y/K), so that as the slope of the k line rises its intersection with the vertical produced from $(\frac{1}{\gamma})^{\frac{1}{\beta}}$ describes the locus of higher and higher equilibria.

⁸ “Economic Growth and Capital Accumulation,” p. 338.

⁹ $\sigma = 1$ is taken to separate “difficult” from “easy” substitution.

It is worthwhile noting that in some cases “plausible” changes in the saving ratio can induce significant changes in equilibrium income growth.

But it is not only equilibrium that matters. In all such processes as these, equilibrium is never literally reached, and when, in equilibrium, some variable such as the capital–labor ratio has to become zero or infinite, it is reasonable to assume that the system will usually take a very long time to get near to equilibrium. In such cases comparisons of equilibrium situations are not particularly useful. However, in the case of Figure 2(d), knowledge of the equilibrium income growth rate does prove useful because it turns out to be the lowest possible income growth rate under those conditions. This can be seen if we substitute (1) in (4) and differentiate y with respect to Y/K .

Then we have

$$\frac{dy}{d(Y/K)} = \gamma s(\beta + 1) \left(\frac{Y}{K}\right) - \gamma n\beta \left(\frac{Y}{K}\right)^{\beta-1},$$

which is positive when $\sigma > 1$ ($0 > \beta > -1$). Hence, y is a monotonically increasing function whose slope is a direct function of s , and the equilibrium income growth rate $y = s(1/\gamma)^{\frac{1}{\beta}}$ is thus the lowest income growth rate which it can attain. Our conclusions about raising the saving ratio then apply *a fortiori* to Figure 2(d).

We are not so fortunate in the case of Figure 2(a), for the function y may or may not have a minimum in the range of attainable values of Y/K . This makes it difficult to offer a general statement about the equilibrium as compared with the nonequilibrium growth rates of income. One would need information about the time path of income in order to be satisfied that a given change in s would make a significant improvement.

Several (rather obvious) qualifications are in order. In the first place, raising the saving ratio may be impossible (without lowering population growth) because no investible surplus above subsistence consumption may exist. Second, it may be very difficult to maintain some of these growth processes and at the same time maintain full employment. In particular the marginal product of capital must become fairly low in Figure 2(d) as the capital–labor ratio gets nearer and nearer to infinity and entrepreneurs’ enthusiasm for investment would undoubtedly dwindle. In Figure 2(a) the marginal product of labor tends to zero, in

which case pressure for a higher real wage could well interfere with full employment of labor.

APPENDIX

- (a) To show that the elasticity of substitution (σ) is a parameter of the production function,

$$Y = [\gamma K^{-\beta} + \mu N^{-\beta}]^{-\frac{1}{\beta}} \quad (\text{A1})$$

such that $\sigma = 1/(1 + \beta)$.

Now

$$\sigma = \frac{(\partial Y/\partial K) \cdot (\partial Y/\partial N)}{Y \cdot (\partial^2 Y/\partial K \partial N)} \quad (\text{A2})$$

when the production function is linear and homogeneous.¹⁰

$$\begin{aligned} \frac{\partial Y}{\partial N} &= \mu \left(\frac{Y}{N}\right)^{\beta+1} \\ \frac{\partial Y}{\partial K} &= \gamma \left(\frac{Y}{K}\right)^{\beta+1} \\ Y \cdot \frac{\partial^2 Y}{\partial K \partial N} &= (\beta + 1) \gamma \left(\frac{Y}{K}\right)^{\beta+1} \mu \left(\frac{Y}{N}\right)^{\beta+1}. \end{aligned}$$

Hence, substituting in (A2) we have

$$\sigma = \frac{1}{1 + \beta}.$$

- (b) The roots of $y - k = \gamma \left(\frac{Y}{K}\right)^\beta k + (1 - \gamma \left(\frac{Y}{K}\right)^\beta) n - k$ will be equilibrium solutions of the system provided they lie within the limits with respect to Y/K imposed by the function. Now $y - k = (k - n)[\gamma (Y/K)^\beta - 1]$; thus, $y = k$ when either $k = n$ or $\gamma (Y/K)^\beta = 1$.

¹⁰ See R. G. D. Allen, *Mathematical Analysis for Economists*, p. 343.

The equilibrium concerned will be stable provided that

$$\frac{dy}{d\left(\frac{Y}{K}\right)} < \frac{dk}{d\left(\frac{Y}{K}\right)} = s.$$

Now

$$\frac{dy}{d(Y/K)} = (\beta + 1)\gamma s \left(\frac{Y}{K}\right)^\beta - \beta\gamma n \left(\frac{Y}{K}\right)^{\beta-1}.$$

If $y = k = n$, $Y/K = n/s$,

$$\frac{dy}{d(Y/K)} = \gamma n^\beta s^{1-\beta},$$

which is stable if $\gamma n^\beta s^{1-\beta} < s$ or $\gamma(n/s)^\beta < 1$. That is, if the golden age falls within achievable values of the output–capital coefficient, it will be stable.

Again if $Y/K = \left(\frac{1}{\gamma}\right)^{\frac{1}{\beta}}$

$$\frac{dy}{d(Y/K)} = s(\beta + 1) - \beta n(\gamma)^{\frac{1}{\beta}}.$$

Thus, this equilibrium is stable if

$$\gamma \left(\frac{n}{s}\right)^{\frac{1}{\beta}} > 1,$$

that is, if it exists.

(c) As $Y/K \rightarrow 0$, $y \rightarrow n$ if $\beta > 0$.

This can be seen if $Y/K = 0$ is substituted in

$$y = \gamma s \left(\frac{Y}{K}\right)^{\beta+1} + \left(1 - \gamma \left(\frac{Y}{K}\right)^\beta\right) n.$$

(d) The propositions contained in (b) and (c) help toward the construction of Figures 2(a), 2(b), 2(c), and 2(d). It remains to show where the y line lies in relation to the k and n lines.

Write

$$y - k = (k - n) \left[\gamma \left(\frac{Y}{K}\right)^\beta - 1 \right],$$

and

$$y - n = (k - n)\gamma \left(\frac{Y}{K}\right)^\beta.$$

If $\gamma (Y/K)^\beta < 1$, that is, $Y/K < (1/\gamma)^{\frac{1}{\beta}}$ if $\beta > 0$ and $Y/K > (1/\gamma)^{\frac{1}{\beta}}$ if $\beta < 0$,
when

$$k < n, y > k, y < n$$

$$k > n, y < k, y > n.$$

(e) The shape of the function

$$\frac{K}{Y} = \left[\frac{1}{\gamma} - \frac{\mu}{\gamma} \left(\frac{N}{Y}\right)^{-\beta} \right]^{-\frac{1}{\beta}} \quad (\text{A3})$$

$$\begin{aligned} \frac{d(K/Y)}{d(N/Y)} &= \frac{-(\mu/\gamma)(N/Y)^{-(\beta+1)} \cdot (K/Y)}{(1/\gamma) - (\mu/\gamma)(N/Y)^{-\beta}} \quad (\text{A4}) \\ &= -\frac{\mu}{\gamma} \left(\frac{K}{N}\right)^{\beta+1} \end{aligned}$$

$$\begin{aligned} \frac{d^2(K/Y)}{d(N/Y)^2} \cdot \frac{1}{[d(K/Y)/d(N/Y)]} &= \frac{d(K/Y)}{d(N/Y)} \cdot \frac{1}{(K/Y)} \\ &\quad -(\beta+1) \left(\frac{N}{Y}\right)^{-1} - \frac{\beta(\mu/\gamma)(N/Y)^{-(\beta+1)}}{(1/\gamma) - (\mu/\gamma)(N/Y)^{-\beta}}, \quad (\text{A5}) \end{aligned}$$

(A4) and (A5) are negative so that $d^2(K/Y)/d(N/Y)^2$ is positive.

Function (A5) thus has a negative slope and is convex to the origin.

When $\sigma > 1$ and $N/Y = 0$, $K/Y = (1/\gamma)^{-\frac{1}{\beta}}$; and when $\sigma > 1$ and $K/Y = 0$, $N/Y = (1/\mu)^{-\frac{1}{\beta}}$.

When $\sigma < 1$ and $N/Y = \infty$, $K/Y = (\frac{1}{\gamma})^{-\frac{1}{\beta}}$; and when $\sigma < 1$ and $K/Y = \infty$, $N/Y = (1/\mu)^{-\frac{1}{\beta}}$.

The Cobb–Douglas function may be seen to be a special case of (A1) for the case $\sigma = 1$, $\beta = 0$, $\gamma + \mu = 1$.

From (A1) (appendix)

$$\begin{aligned}\log Y &= \frac{-\log[\gamma K^{-\beta} + \mu N^{-\beta}]}{\beta} \\ \text{Lt.}_{\beta \rightarrow 0} \log Y &= -\text{Lt.}_{\beta \rightarrow 0} \frac{\frac{d}{d\beta} \log[\gamma K^{-\beta} + \mu N^{-\beta}]}{d(\beta)/d\beta} \\ &= -\text{Lt.}_{\beta \rightarrow 0} - \frac{\gamma K^{-\beta} \log K - \mu N^{-\beta} \log N}{\gamma K^{-\beta} + \mu N^{-\beta}} \\ &= \gamma \log K + \mu \log N.\end{aligned}$$

Hence,

$$Y = K^\gamma N^\mu. \quad (\text{A6})$$

Now rearranging we have

$$\begin{aligned}\frac{K}{Y} &= \left(\frac{N}{Y}\right)^{-\frac{\mu}{\gamma}} \\ \frac{d(K/Y)}{d(N/Y)} &= -\frac{\mu}{\gamma} \left(\frac{N}{Y}\right)^{-\left(\frac{\mu}{\gamma}+1\right)},\end{aligned}$$

which is less than zero, and

$$\frac{d^2(K/Y)}{d(N/Y)^2} = \left(\frac{\mu}{\gamma}\right) \left(\frac{\mu}{\gamma} + 1\right) \left(\frac{N}{Y}\right)^{-\left(\frac{\mu}{\gamma}+2\right)},$$

which is positive.

It can also be seen that this function will be asymptotic to both axes.