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169. Quantum Stochastic Processes and Noncommutative Geometry. By KALYAN B. SINHA and DEBASHISH GOSWAMI

QUANTUM STOCHASTIC PROCESSES AND NONCOMMUTATIVE GEOMETRY

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Preface

On the one hand, in almost all the scientific areas, from physical to social sciences, biology to economics, from meteorology to pattern recognition in remote sensing, the theory of classical probability plays a major role and on the other much of our knowledge about the physical world at least is based on the quantum theory [12]. In a way, quantum theory itself is a new kind of theory of probability (in the language of von Neumann and Birkhoff) (see for example [106]) which contains the classical model, and therefore it is natural to extend the other areas of classical probability theory, in particular the theory of Markov processes and stochastic calculus to this quantum model.

There are more than one possible ways (see for example [127]) to construct the above-mentioned extension and in this book we have chosen the one closest to the classical model in spirit, namely that which contains the classical theory as a submodel. This requirement has ruled out any discussion of areas such as free and monotone-probability models. Once we accept this quantum probabilistic model, the 'grand design' that engages us is the 'canonical construction of a $*$-homomorphic flow (satisfying a suitable differential equation) on a given algebra of observables such that the expectation semigroup is precisely the given contractive semigroup of completely positive maps on the said algebra'.

This problem of 'dilation' is here solved completely for the case when the semigroup has a bounded generator, and also for the more general case (of an unbounded generator) with certain additional conditions such as symmetry and/or covariance with respect to a Lie group action. However, a certain amount of space has to be devoted to develop the needed techniques and structures, and the reader is expected to be well equipped with the basics of functional analysis, theory of Hilbert spaces and of operators in them and of probability theory in order to master these.

A beginner with the above-mentioned background may read Chapters 1 to 6 at first and may leave the rest for a second reading. In some places, mathematical assertions have been made without proof wherever we felt that the proof is essentially similar to a detailed proof of an earlier statement or when the verification of the same can be left as an exercise.

Due to lack of space, not all equations have been displayed and long expressions had to be broken at the end of a line, any inconvenience due to this is regretted. The open square symbol denotes the end of a proof. The reference list is far from complete, we have often included only a recent or a representative paper. We apologize for any unintended exclusion of a reference.

It is a pleasure to remember here people who have contributed to the preparation of this book. Professor K. R. Parthasarathy was instrumental in introducing us to the subject and one of us (K. B. S.) has collaborated with him extensively over nearly two decades; without the insights and masterly expositions of him and of Professor P. A. Meyer, the subject may not have reached the stage it is in now. We thank all our friends, collaborators and members of the Q–P club who have helped us directly or indirectly in this endeavor. In particular, we must mention Professors Luigi Accardi, Robin Hudson, V. P. Belavkin, Martin Lindsay, Franco Fagnola, Stephane Attal, Jean-Luc Sauvageot, Burkhard Kümmerer, Hans Maassen, Rajarama Bhat and Dr Arup Pal and Dr Partha Sarathi Chakraborty. We are grateful to the Indian Statistical Institute (both Delhi and Kolkata campuses) for providing the necessary facilities, Indo-French Centre for the Promotion of Advanced Research and DST-DAAD agencies for making many collaborations possible. One of us (D. G.) would like to thank the Alexander von Humboldt Foundation for a postdoctoral fellowship during 2000–01 (and also later visits under its scheme of 'resumption of fellowship'), when part of the work covered by this book was done. We must also thank Dr Lingraj Sahu, who as a graduate student at a critical stage of writing the monograph, helped with introduction of a part of the material and Mr Joydip Jana for help with proofreading. One of the authors (D. G.) dedicates this book to his wife, Gopa and the youngest addition to his family, expected possibly before this book sees the light of the day; and acknowledges with gratitude the constant encouragement and support from his parents, mother-in-law and Amit-da during the writing of the book.

As is often the case in any such enterprise, some important topics (e.g. stop times) have been left out. The responsibility for the choice of topics as well as for any omissions and shortcomings of the text is entirely ours. We can only hope that this monograph will enthuse some researchers and students to solve some of the problems left unsolved.

K. B. Sinha
Debashish Goswami

Notation

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}_+$	The set of nonnegative real numbers
$\mathbb{Z}$	The set of integers
S^1	The circle group
$\mathbb{T}^n$	The n-torus
C	The set of bounded continuous functions in $L^2(\mathbb{R}_+, k_0)$ or $L^2 \cap L^4_{\mathrm{loc}}(\mathbb{R}_+, k_0)$
$\mathrm{Dom}(T)$	Domain of an operator T
$\mathrm{Ran}(T)$	Range of T
$\mathrm{Ker}(T)$	Kernel of T
$\mathrm{Sp}(A)$	Complex linear span of vectors in the set A
$\dim(V)$	Dimension of the vector space V
$\mathrm{Im}(x), \mathrm{Re}(x)$	Imaginary and real parts of x (complex number or bounded operator)
$h, \mathcal{H}$	Hilbert spaces
$\mathrm{Lin}(V_1, V_2)$	Set of linear maps from V_1 to V_2 (vector spaces).
$\mathcal{B}(\mathcal{H})$	The set of bounded linear operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}(\mathcal{H}, \mathcal{K})$	The set of bounded linear maps from $\mathcal{H}$ to $\mathcal{K}$ (Hilbert spaces)
$\mathcal{B}^{\mathrm{s.a.}}(\mathcal{H})$	The real Banach space of all bounded self-adjoint operators on $\mathcal{H}$
$\mathcal{K}(\mathcal{H})$	The set of compact operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}_1(\mathcal{H})$	The complex Banach space of trace-class operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}_1^{\mathrm{s.a.}}(\mathcal{H})$	The real Banach space of self-adjoint trace-class operators on $\mathcal{H}$
$\mathcal{L}(E, F)$	The set of adjointable maps from E to F (Hilbert modules)
$\mathcal{L}(E)$	The set of adjointable maps on a Hilbert module E

$\mathcal{K}(E)$	The set of compact adjointable maps on a Hilbert module E
$\mathcal{A}'$	The commutant of (C^* or von Neumann algebra) $\mathcal{A}$
$\mathcal{A}_+$	The set of positive elements of C^* or von Neumann algebra $\mathcal{A}$.
$\mathcal{M}(\mathcal{A})$	The multiplier algebra of $\mathcal{A}$
$\Omega, \Omega_{\mathcal{A}}$	The set of all states, of all normal linear functional on a C^*- or von Neumann algebra $\mathcal{A}$
$\otimes_{\text{alg}}$	Algebraic tensor product (between spaces or algebras)
$\Gamma(\mathcal{H})$	The symmetric Fock space over the Hilbert space $\mathcal{H}$
Γ	The symmetric Fock space over $L^2(\mathbb{R}_+, k_0)$ for some Hilbert space k_0
$\Gamma^f(\mathcal{H})$	The free Fock space over $\mathcal{H}$
k_t	The Hilbert space $L^2([0,t], k_0)$ for some Hilbert space k_0
k^t	$L^2((t,\infty), k_0)$
Γ_t	$\Gamma(k_t)$
Γ^t	$\Gamma(k^t)$
$e(f)$	The exponential vector $\oplus_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}}$
$\Gamma(A)$	The second quantization of A
χ_A	The characteristic function of the set A
f_t	The function $f\chi_{[0,t]}$
f^t	The function $f\chi_{(t,\infty)}$
Θ	The structure matrix
$\mathcal{U}_t$	Time reversal operator in Fock space
$\mathcal{A} >\!\!\lhd_\alpha G$	The crossed product of $\mathcal{A}$ (C^* or von Neumann algebra) by the action α of a group G
$a_R^\dagger(\cdot), a_\delta^\dagger(\cdot)$	The creation integrator processes associated with operator R and map δ respectively
$a_R(\cdot), a_\delta(\cdot)$	The annihilation integrator processes associated with operator R and map δ respectively
$\Lambda_T(\cdot), \Lambda_\sigma(\cdot)$	The number integrator processes associated with operator T and map σ respectively
$\mathcal{I}_{\mathcal{L}}(\cdot)$	The time integrator process associated with map $\mathcal{L}$ on $\mathcal{A}$
L^p_{loc}	The set of all $f \in L^2(\mathbb{R}_+, k_0)$ such that $\int_0^t \|f(s)\|^p ds < \infty$ for every $t \geq 0$
$L^p(\mathcal{A}, \tau)$	The noncommutative L^p spaces associated with trace τ