1
Introduction

The motivations for writing the present monograph are three-fold: firstly from a physical point of view and secondly from two related, but different mathematical angles.

At the present time our mathematical understanding of a conservative quantum mechanical system is reasonably complete, both from the direction of a consistent abstract theory as well as from the one of mathematical theories of applications in many explicit physical systems like atoms, molecules etc. (see for example the books [12] and [108]). However, a nonconservative (open/dissipative) quantum mechanical system does not enjoy a similar status. Over the last seven decades there have been many attempts to make a theory of open quantum systems beginning with Pauli [104]. Some of the typical references are: Van Hove [126], Ford *et al.* [52], along with the mathematical monograph of Davies [35]. The physicists' Master equation (or Langevin equation) was believed to describe the evolution of a nonconservative open quantum (or classical) mechanical system, a mathematical description of which can be found in Feller's book [50].

Physically, one can conceive of an open system as the 'smaller subsystem' of a total ensemble in which the system is in interaction with its 'larger' environment (sometimes called the bath or reservoir). The total ensemble with a very large number of degrees of freedom undergoes (conservative) evolution, obeying the laws of standard quantum mechanics. However, for various reasons, practical or otherwise, it is of interest only to observe the system and not the reservoir, and this 'reduced dynamics' in a certain sense obeys the Master equation (for a more precise description of these, see [35]). Since it is often impossible and impractical to solve the equation of evolution of the total ensemble, it is often meaningful to replace the reservoir by a 'suitable stochastic process' and couple the system with the stochastic process. In the case in which the stochastic process is classical, the total evolution can be described by a suitable stochastic differential equation (for an introduction to this, the reader is referred to [75] and [41]). The standard Langevin equation [52] involving the stochastic process should restore the conservativeness of the total system albeit for almost all paths. However, in many of the models studied by physicists this is not so.

The simplest quantum mechanical system is the so-called harmonic oscillator. However, the (sub-critically) damped harmonic oscillator which has been studied in classical physics since the time of Newton eludes a consistent treatment in conventional quantum mechanics. In the view of the present authors, this happens because the damped harmonic oscillator is a nonconservative, dissipative system and cannot be understood as a flow in a symplectic manifold (classical case) or in a standard Weyl canonical commutation relations (CCR) algebra (quantum case). One possible way to model this is to represent the environment or reservoir (responsible for the friction or the damping term) by an appropriate stochastic process, restore the unitary stochastic evolution of the quantum system and then project back to the 'system space' by 'washing out' the influence of the stochastic process (taking expectation with respect to the stochastic part) to get back the required nonconservative dynamics. This has been studied in [119] and has also been described in some detail in Chapter 7. Thus one can enunciate a philosophy, not too far away from that of the physicists, that given a nonconservative dynamics of a quantum system, one aim is to canonically construct the stochastic process which will represent the environment so that the two together undergo a conservative evolution and the projection to the system space restores exactly the nonconservative evolution. There is a further aim of the physicist, viz. to obtain the stochastic process mentioned above in a suitable approximation from the mechanical descriptions of the particles constituting the reservoir and of their interactions with the observed system. This aspect is not treated in this monograph and the reader is referred to [4], [8] and [35].

There is an exact mathematical counterpart to the picture in physicists' mind as described above. Given a finite probability space $S \equiv \{1, 2, \ldots, n\}$ with probability distribution given by the vector $p \equiv (p_1, p_2, \ldots, p_n)$ on it and a stochastic (or Markov) matrix $(t_{ij})_{ij=1}^n$ such that $t_{ij} \geq 0$, $\sum_{j=1}^n t_{ij} = 1$, one can associate a (discrete) evolution $(Tf)(i) = \sum_{j=1}^n t_{ij} f(j)$ with $f : \{1, 2, \ldots, n\} \to \mathbb{R}$. Then one observes that

(i) T maps positive functions f to positive functions and maps identity function to itself.
(ii) The probability distribution vector p is in one-to-one correspondence with the dual ϕ_p of the algebra of functions on S by $p \mapsto \phi_p$, where

$\phi_p(f) = \sum_{i=1}^n p_i f(i)$, and this induces a dual dynamics T^* given by $(T^*\phi_p)(\chi_j) = \sum_{i=1}^n p_i t_{ij}$, where χ_k denotes the characteristic function of the singleton set $\{k\}$.

(iii) $T^n, n = 0, 1, \ldots,$ and $T^{*n}, n = 0, 1, 2 \ldots$ provide two discrete (dynamical) semigroups, the second being dual to the first; and clearly T^n for each n satisfies the property (i).

There is a standard construction of a Markov process (in this case Markov chain); see e.g. Feller's book [50]. This procedure extends naturally, beginning with the consideration of the algebra of functions on S as the algebra of $n \times n$ diagonal matrices and $\{T^n\}_{n=0,1,2,\ldots}$ as a positive semigroup on that, to the more general picture considering semigroups (discrete or continuous parameter) on the noncommutative algebra of all $n \times n$ matrices. What is perhaps surprising and is contrary to intuition in classical probability is that a very large class of Markov processes (including Markov chains) can be described by quantum stochastic differential equations in Fock space, again facilitating many computations ([99, 100]).

At this point an important generalization of the class of positive maps on an algebra makes its entrance. From a physical point of view, consider the following scenario. Let $\mathcal{H}$ be the Hilbert space of a localized quantum system A in a box and let there exist another quantum system B with associated Hilbert space $\mathbb{C}^n$. The system B is so far removed from A that there is no interaction between A and B and thus the Hilbert space for the joint system A and B is $\mathcal{H} \otimes \mathbb{C}^n$. Let T_n be the positive linear map which describes an operation on the joint system that does not affect B, given by $T_n(x \otimes y) = T(x) \otimes y$ for $x \in \mathcal{B}(\mathcal{H})$, $y \in \mathcal{B}(\mathbb{C}^n)$ (here $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on the Hilbert space $\mathcal{H}$ defined everywhere) for some positive linear map T on $\mathcal{B}(\mathcal{H})$. It seems reasonable to expect that given a positive linear map T on $\mathcal{B}(\mathcal{H})$, it should be such that for every natural number n, T_n given above should be positive. In such a case, T is said to be completely positive (CP) and such CP maps or semigroups of such maps play a very important role in the description of nonconservative dynamics on quantum systems. It is also useful to note that if the algebra involved is commutative (like the algebra of $n \times n$ diagonal matrices in the first example instead of $\mathcal{B}(\mathcal{H})$ or the whole matrix algebra) positivity and complete positivity are equivalent and that is why complete positivity does not surface in the context of nonconservative evolutions of classical physical systems. A detailed mathematical study of CP maps and of semigroups of CP maps on an algebra is done in Chapters 2 and 3, respectively.

As we had mentioned earlier in the context of a physical subsystem interacting with a reservoir in such a way that the reduced dynamics is governed by a

Master equation, it is natural to assume that the Master equation is just the differential form of a contractive semigroup of CP maps on the algebra describing the subsystem. Now we can turn this into a very interesting (and demanding) mathematical question: does there exist a 'suitable' probabilistic model for (a) the reservoir and for (b) its interaction with the given subsystem such that the expectation of the total evolution with respect to the probabilistic variables give the CP semigroup we started with? This is the general problem of 'dilation of a contractive semigroup of CP maps on a given algebra'. This problem is solved in Chapter 6 in complete generality under the hypotheses that the given semigroup of CP maps is uniformly continuous so that its generator acting on the given algebra is bounded.

There are complete descriptions of the structure of the generator of a uniformly continuous semigroup of CP maps on an algebra in the third chapter. Unfortunately the situation is far from settled for a similar question if the semigroup is only strongly continuous, which is, as is often the case, more interesting from the point of view of applications. However, if we pretend that the generator of the strongly continuous semigroup of CP maps on the algebra formally looks similar to that for the uniformly continuous case, then under certain hypotheses a class of strongly continuous semigroups can be constructed such that its generator coincides with the formal one on suitable domains. This is described in the second section of the same chapter along with an applications to a large class of classical Markov processes and also to the irrational rotation algebra which is a type Π_1 factor von Neumann algebra. More details on these constructions and results on the unital nature of the semigroups, so constructed, can be found in Chebotarev [25]. This chapter ends with an important abstract theorem on noncommutative Dirichlet forms associated with a strongly continuous semigroup of CP maps on a von Neumann algebra equipped with a normal faithful semifinite trace. This result is then used in Chapter 8 to solve the dilation problem for such semigroups.

In order to carry out the program charted out in an earlier paragraph, it is necessary to develop some language and machinery. In Chapter 4, the basic theories of Hilbert C^*- and von Neumann modules and of group actions on them are presented. These ideas are then used to develop an elaborate theory of stochastic integration and quantum Itô formulae in symmetric Fock spaces extending the earlier theory as described in [97]. This language seems to be sufficiently powerful to allow a large class of unbounded operator-valued processes in Fock space to be treated. These methodologies were then used to solve Hudson–Parthasarathy (H–P)-type stochastic quantum differential equations with bounded coefficients (Chapter 5) and with unbounded coefficients (Chapter 7) giving unitary or isometric evolutions in a suitable Hilbert space as solutions. The Evans–Hudson (E–H)-type equation of observable or of an element of an algebra is re-interpreted as an equation on the space of maps on a suitable Fock Hilbert module and for bounded coefficient case, such equations are solved in Chapter 5. This language and associated machinery are important because they allow us to answer in the affirmative the problem of the dilation of a uniformly continuous semigroup of CP maps on an algebra.

Chapter 6 uses the tools of Chapters 4 and 5 to show that given a uniformly continuous semigroup of CP maps on a von Neumann algebra, there exists a quantum probabilistic model in the Fock space such that there is a E–H-type quantum stochastic differential equation describing the stochastic evolution of the observable algebra of the quantum subsystem coupled to the quantum stochastic process in Fock space modeling the reservoir, and such that the expectation gives back the original CP semigroup. This construction is canonical and interestingly gives a quantum stochastic differential equation for the evolution so that further computations for any other observable effects may be facilitated.

The mathematical problem of stochastic dilation of a semigroup of CP maps on a C^*- or von Neumann algebra, uniformly or strongly continuous, with the additional requirement that the dilated map on the algebra satisfies a quantum stochastic differential equation in Fock space and is a $*$-homomorphism on the algebra of observables is the central mathematical problem treated in this book. The property of $*$-homomorphism of such maps is a basic requirement of any quantum theory and the fact that these also satisfy a differential equation makes the family of dilated maps a stochastic flow of $*$-homomorphisms on the algebras. In fact, Chapters 6 and 8 are devoted to the final steps of the solution of this problem, the first for the uniformly continuous semigroup and the second for the strongly continuous one, while the Chapters 2 to 5 and Chapter 7 deal with preliminary materials and develop the machinery needed. This completes our discussions on the central mathematical problem treated here along with its connection to applications, arising from the physics of open quantum systems.

There is a another mathematical direction from which we approach the central mathematical problem of stochastic dilation, viz. that of noncommutative geometry. Chapter 9 should not be and cannot be thought of as an exposition on the rapidly developing subject of noncommutative geometry as created by Alain Connes [28] (the reader may also look at the books [82] and [56]). Instead, after some introduction to basic concepts in differential geometry and elements of noncommutative geometry, three explicit examples are worked out and in each case an appropriate associated stochastic process (classical or quantum) is constructed. Much more study in these areas remains to be done; for example one can investigate whether the nontrivial curvature in the Quantum Heisenberg manifold can be captured in terms of the stochastic processes on it.

We think the spirit of the book is perhaps well-described in the preface by Luigi Accardi in *Probability Towards 2000* [3] and we quote:

> The reason why the interaction of probability with quantum physics is different from the above mentioned ones is that the problem here is not only to apply classical techniques or to extend them to situations which, being even more general, still remain within the same qualitative type of intuition, language and techniques. Furthermore, the formalism of quantum theory, with its complex wave functions and Hilbert spaces, operators instead of random variables, creates a distance between the mathematical model and the physical phenomena which is certainly greater than that of classical physics. For these reasons, these new languages and techniques might be perceived as extraneous by some classical probabilists and researchers in mathematical statistics. However, the developments motivated by quantum theory provide not only powerful theoretical tools to probability, but also some conceptual challenges which can enter into the common education of all mathematicians in the same way as happened for the basic qualitative ideas of non-Euclidean geometries.

2
Preliminaries

In this chapter we shall introduce all the basic materials and preliminary notions needed later on in this book.

2.1 C^* and von Neumann algebras

For the details on the material of this section, the reader may be referred to [125], [40] and [76].

2.1.1 C^*-algebras

An abstract normed $*$-algebra $\mathcal{A}$ is said to be a *pre-C^*-algebra* if it satisfies the C^*-property : $\|x^*x\| = \|x\|^2$. If $\mathcal{A}$ is furthermore complete under the norm topology, one says that $\mathcal{A}$ is a *C^*-algebra*. The famous structure theorem due to Gelfand, Naimark and Segal (GNS) asserts that every abstract C^*-algebra can be embedded as a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ (the set of all bounded linear operators on some Hilbert space $\mathcal{H}$). In view of this, we shall fix a complex Hilbert space $\mathcal{H}$ and consider a concrete C^*-algebra $\mathcal{A}$ inside $\mathcal{B}(\mathcal{H})$. The algebra $\mathcal{A}$ is said to be *unital* or *nonunital* depending on whether it has an identity or not. However, even any nonunital C^*-algebra always has a net (sequence in case the algebra is separable in the norm topology) of *approximate identity*, that is, an nondecreasing net e_μ of positive elements such that $e_\mu a \to a$ for all $a \in \mathcal{A}$. Note that the set of compact operators on an infinite dimensional Hilbert space $\mathcal{H}$, to be denoted by $\mathcal{K}(\mathcal{H})$, is an example of nonunital C^*-algebra.

We now briefly discuss some of the important aspects of C^*-algebra theory. First of all, let us mention the following remarkable characterization of commutative C^*-algebras.

Theorem 2.1.1 *Every commutative C^*-algebra $\mathcal{A}$ is isometrically isomorphic to the C^*-algebra $C_0(X)$ consisting of complex-valued functions on a locally compact Hausdorff space X vanishing at infinity. In case $\mathcal{A}$ is unital, X is compact.*

If $\mathcal{A}$ is nonunital, there is a canonical method of adjoining an identity so that $\mathcal{A}$ is embedded as an ideal in a bigger unital C^*-algebra $\hat{\mathcal{A}}$. In view of this, let us assume $\mathcal{A}$ to be unital for the rest of the subsection, unless otherwise mentioned. For $x \in \mathcal{A}$, the *spectrum* of x, denoted by $\sigma(x)$, is defined as the complement of the set $\{z \in \mathbb{C} : (z1 - x)^{-1} \in \mathcal{A}\}$. It is known that for a self-adjoint element x, $\sigma(x) \subseteq \mathbb{R}$, and moreover, a self-adjoint element x is *positive* (that is, x is of the form y^*y for some y) if and only if $\sigma(x) \subseteq [0, \infty)$. There is a rich functional calculus which enables one to form functions of elements of the C^*-algebra. For any complex function which is holomorphic in some domain containing $\sigma(x)$, one obtains an element $f(x) \in \mathcal{A}$ by the holomorphic functional calculus. Furthermore, for any *normal* element x (that is, $xx^* = x^*x$), there is a continuous functional calculus sending $f \in C(\sigma(x))$ to $f(x) \in \mathcal{A}$ where $f \mapsto f(x)$ is a $*$-isometric isomorphism from $C(\sigma(x))$ onto $C^*(x)$, the sub C^*-algebra of $\mathcal{A}$ generated by x. In particular, for any positive element x, we can form a positive *square root* $\sqrt{x} \in \mathcal{A}$ satisfying $\sqrt{x}^2 = x$. For any element $x \in \mathcal{A}$, we define its *absolute value*, denoted by $|x|$, to be the element $\sqrt{x^*x}$. The *real and imaginary parts* of x, denoted by $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ respectively, are defined by, $\mathrm{Re}(x) = (x + x^*)/2$, $\mathrm{Im}(x) = (x - x^*)/2i$, so that we have, $x = \mathrm{Re}(x) + i\mathrm{Im}(x)$. For a self-adjoint element x, we define two positive elements x^+ and x^-, called respectively the *positive and negative parts* of x, by setting $x^+ = (x + |x|)/2$, $x^- = (|x| - x)/2$. Clearly, x can be decomposed as $x = x^+ - x^-$ and furthermore $x^+x^- = 0$. A linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is said to be *positive* if $\phi(x^*x) \geq 0$ for all x. It is a useful result that an element $x \in \mathcal{A}$ is positive if and only if $\phi(x) \geq 0$ for every positive functional ϕ on $\mathcal{A}$. It can be shown that the algebraic property of positivity implies the boundedness of ϕ, in particular $\|\phi\| = \phi(1)$. Any positive linear functional ϕ with $\phi(1) = 1$ is called a *state* on $\mathcal{A}$. The set of all states is a convex set which is compact in the weak-$*$ topology, hence it has extreme points, called *pure states*, and the set of states is obtained as the closed convex hull of the pure states. A state ϕ is said to be a *trace* if $\phi(ab) = \phi(ba)$ for all $a, b \in \mathcal{A}$. It is said to be *faithful* if $\phi(x^*x) = 0$ implies $x = 0$. The following result, known as the *GNS construction* for a state, is worthy of mention.

Proposition 2.1.2 *Given a state ϕ on $\mathcal{A}$, there exists a triple (called the GNS triple) $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$, consisting of a Hilbert space $\mathcal{H}_\phi$, a $*$-representation π_ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\phi)$ and a vector $\xi_\phi \in \mathcal{H}_\phi$ which is cyclic in the sense that*

$\{\pi_\phi(x)\xi_\phi : x \in \mathcal{A}\}$ is total in $\mathcal{H}_\phi$, satisfying

$$\phi(x) = \langle \xi_\phi, \pi_\phi(x)\xi_\phi \rangle.$$

Moreover, ϕ is pure if and only if π_ϕ is irreducible.

We shall need to extend the scope of the GNS construction to the case of densely defined positive functionals, at least for *semifinite, faithful, positive traces*, which we discuss now. Let us denote by $\mathcal{A}_+$ the set of positive elements of $\mathcal{A}$. Let $\tau : \mathcal{A}_+ \to [0, \infty]$ be a map satisfying $\tau(a + b) = \tau(a) + \tau(b)$, $\tau(\lambda a) = \lambda\tau(a)$ and $\tau(aa^*) = \tau(a^*a)$ for $a, b \in \mathcal{A}_+, \lambda \in \mathbb{R}_+$. Assume furthermore that $\mathcal{I} \equiv \{a \in \mathcal{A} : \tau(a^*a) < \infty\}$ is norm-dense in $\mathcal{A}$ and $\tau(a^*a) = 0$ implies $a = 0$. Such a map τ is called a semifinite, faithful, positive trace, and it can be uniquely extended to the dense subspace (in fact a both-sided ideal) $\mathcal{I}$ as a linear functional, also denoted by τ. The GNS construction can be generalized to such a trace in the following sense.

Proposition 2.1.3 *There exists a Hilbert space $\mathcal{H}$, a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a linear map $\eta : \mathcal{I} :\to \mathcal{H}$, such that $\tau(a^*bc) = \langle \eta(a), \pi(b)\eta(c) \rangle$ for all $a, c \in \mathcal{I}, b \in \mathcal{A}$. Furthermore, the range of η is dense in $\mathcal{H}$. Such a triple $(\mathcal{H}, \pi, \eta)$ is unique in the sense that for any other such triple $(\mathcal{H}', \pi', \eta')$, we can find a unitary $\Gamma : \mathcal{H} \to \mathcal{H}'$ such that $\pi'(b) = \Gamma\pi(b)\Gamma^*$ and $\eta'(a) = \Gamma\eta(a)$ for $a \in \mathcal{I}, b \in \mathcal{A}$.*

We shall denote the Hilbert space (unique upto identification) $\mathcal{H}$ in the above proposition by $L^2(\mathcal{A}, \tau)$ or simply $L^2(\tau)$ if $\mathcal{A}$ is understood from the context. It can be shown that $\pi(b)\eta(a) = \eta(ba)$, and thus, if $\mathcal{A}$ is unital and $1 \in \mathcal{I}$, which is equivalent to the boundedness of τ, we have a cyclic vector $\eta(1)$.

For a nonunital C^*-algebra, we say that a positive functional ϕ on $\mathcal{A}$ is a state if $\lim \phi(e_\mu) = 1$ for any approximate identity e_μ of $\mathcal{A}$. A positive element $a \in \mathcal{A}$ is said to be *strictly positive* if $\phi(a)$ is nonzero for every state ϕ on $\mathcal{A}$. It is known that $b \in \mathcal{A}_+$ is strictly positive if and only if $b\mathcal{A}_+ := \{ba, a \in \mathcal{A}_+\}$ is norm-dense in $\mathcal{A}_+$.

We shall conclude the discussion on C^*-algebras with the definition of *multiplier algebra*. For a C^*-algebra $\mathcal{A}$ (possibly nonunital), its multiplier algebra, denoted by $\mathcal{M}(\mathcal{A})$, is defined as the maximal C^*-algebra which contains $\mathcal{A}$ as an *essential two-sided ideal*, that is, $\mathcal{A}$ is an ideal in $\mathcal{M}(\mathcal{A})$ and for $y \in \mathcal{M}(\mathcal{A})$, $ya = 0$ for all $a \in \mathcal{A}$ implies $y = 0$. In case $\mathcal{A}$ is unital, one has $\mathcal{M}(\mathcal{A}) = \mathcal{A}$ and for $\mathcal{A} = C_0(X)$ where X is a noncompact, locally compact Hausdorff space, $\mathcal{M}(\mathcal{A}) = C(\hat{X})$, where $\hat{X}$ denotes the Stone–Čech compactification of X. The norm of $\mathcal{M}(\mathcal{A})$ is given by $\|x\| := \sup_{a \in \mathcal{A}, \|a\| \le 1}\{\|xa\|, \|ax\|\}$. Furthermore, there is a canonical locally convex topology, called the *strict topology* on $\mathcal{M}(\mathcal{A})$, which is given by the family of seminorms $\{\|.\|_a, a \in \mathcal{A}\}$, where $\|x\|_a := \text{Max}(\|xa\|, \|ax\|)$, for $x \in \mathcal{M}(\mathcal{A})$. We say that an embedding $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is *nondegenerate* if for $u \in \mathcal{H}$, $au = 0$ for all $a \in \mathcal{A}$ implies that $u = 0$. It is possible to show by simple arguments that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is nondegenerate if and only if $\{au, a \in \mathcal{A}, u \in \mathcal{H}\}$ is total in $\mathcal{H}$. Given a nondegenerate embedding $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, we have that $\mathcal{M}(\mathcal{A}) \cong \{x \in \mathcal{B}(\mathcal{H}) : xa, ax \in \mathcal{A}, \ \text{for all } a \in \mathcal{A}\}$.

2.1.2 von Neumann algebras

As a Banach space, $\mathcal{B}(\mathcal{H})$ is equipped with the operator-norm topology, but there are other important and interesting topologies that can be given to it, making it a locally convex (but not normable in general) topological space. The most useful ones are weak, strong, ultra-weak and ultra-strong topologies. However, although $\mathcal{B}(\mathcal{H})$ is complete in each of these topologies, a unital sub C^*-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ need not be so. It can be shown that $\mathcal{A}$ is complete in all of the above four locally convex topologies if and only if it is complete in any one of them, and in such a case $\mathcal{A}$ is said to be a *von Neumann algebra*. Furthermore, the strong (respectively, weak) and ultra-strong (respectively, ultra-weak) topologies coincide on norm-bounded convex subsets of $\mathcal{A}$. It is known that if $\mathcal{H}$ is separable, then any norm-bounded subset of $\mathcal{A}$ is metrizable in each of the ultra-weak and ultra-strong topologies. The natural notion of isomorphism between two von Neumann algebras is an algebraic $*$-isomorphism which is also a homeomorphism of the respective ultra-weak topologies. However, there is a stronger notion, called *spatial isomorphism*. Two von Neumann algebras $\mathcal{A}_1 \subseteq \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{A}_2 \subseteq \mathcal{B}(\mathcal{H}_2)$ are said to be *spatially isomorphic* if there is a unitary operator U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $U^*\mathcal{A}_2U = \mathcal{A}_1$.

The following theorem, known as the *Double commutant theorem* due to von Neumann is of fundamental importance in the study of von Neumann algebras. Note that for any subset $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$, we denote by $\mathcal{B}'$ the commutant of $\mathcal{B}$, that is, $\mathcal{B}' = \{x \in \mathcal{B}(\mathcal{H}) : xb = bx \ \text{ for all } \ b \in \mathcal{B}\}$.

Theorem 2.1.4 *A unital $*$-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''(\equiv (\mathcal{A}')')$.*

For the rest of this subsection, let us denote by $\mathcal{A}$ a unital von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. $\mathcal{A}$ is said to be *σ-finite* if there does not exist any uncountable family of mutually orthogonal nonzero projections in $\mathcal{A}$.

We say that $\mathcal{A}$ is a *factor* if the center is trivial, that is, $\mathcal{A} \cap \mathcal{A}' = \{\lambda 1, \lambda \in \mathbb{C}\}$. The importance of factors stems from the result (see [40]) that an arbitrary von Neumann algebra can be decomposed in a suitable technical sense as a 'direct integral' of factors. A factor $\mathcal{A}$ is called *hyperfinite* if there is an increasing sequence of finite dimensional $*$-subalgebras, say $\mathcal{A}_n$, $n = 1, 2, \ldots$, of $\mathcal{A}$ such