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Barotropic geophysical flows and two-dimensional fluid flows: elementary introduction

1.1 Introduction

The atmosphere and the ocean are the two most important fluid systems of our planet. The bulk of the atmosphere is a thin layer of air 10 km thick that engulfs the earth, and the oceans cover about 70% of the surface of our planet. Both the atmosphere and the ocean are in states of constant motion where the main source of energy is supplied by the radiation of the sun. The large-scale motions of the atmosphere and the ocean constitute geophysical flows and the science that studies them is geophysical fluid dynamics. The motions of the atmosphere and the ocean become powerful mechanisms for the transport and redistribution of energy and matter. For example, the motion of cold and warm atmospheric fronts determine the local weather conditions; the warm waters of the Gulf Stream are responsible for the temperate climate in northern Europe; the winds and the currents transport the pollutants produced by industries. It is clear that the motions of the atmosphere and the ocean play a fundamental role in the dynamics of our planet and greatly affect the activities of mankind.

It is apparent that the dynamical processes involved in the description of geophysical flows in the atmosphere and the ocean are extremely complex. This is due to the large number of physical variables needed to describe the state of the system and the wide range of space and time scales involved in these processes. The physical variables may include the velocity, the pressure, the density, and, in addition, the humidity in the case of atmospheric motions or the salinity in the case of oceanic motions. The physical processes that determine the evolution of the geophysical flows are also numerous. They may include the Coriolis force due to the earth's rotation; the sun's radiation; the presence of topographical barriers, as represented by mountain ranges in the case of atmospheric flows and the ocean floor and the continental masses in the case of oceanic flows. There may be also dissipative energy mechanisms, for example due to eddy diffusivity or Ekman drag. The ranges of spatial and temporal scales involved in the description of geophysical flows is also very large. The space scales may vary from a few hundred meters to thousands of kilometers. Similarly, the time scales maybe as short as minutes and as long as days, months, or even years.

The above remarks make evident the need for simplifying assumptions regarding the relevant physical mechanisms involved in a given geophysical flow process, as well as the relevant range of space and time scales needed to describe the process. The treatises of Pedlosky (1987) and Gill (1982) are two excellent references to consult regarding the physical foundations of geophysical flows and different simplifying approximations utilized in the study of the various aspects of geophysical fluids. Here we concentrate on large-scale flows for the atmosphere or *mesoscale* flows in the oceans. The simplest set of equations that meaningfully describes the motion of geophysical flows under these circumstances is given by the:

Barotropic quasi-geostrophic equations

$$\begin{aligned}
&\frac{Dq}{Dt} = \mathcal{D}(\Delta)\psi + \mathcal{F}(\vec{x}, t) \\
&q = \omega + \beta y + h(x, y), \text{ where } \omega = \Delta\psi \\
&\vec{v} = \nabla^{\perp}\psi = \begin{pmatrix} -\dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\psi}{\partial x} \end{pmatrix},
\end{aligned} \tag{1.1}$$

where $\frac{D}{Dt}$ stands for the advective (or material) derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$$

and Δ denotes the Laplacian operator

$$\Delta = \operatorname{div} \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In equation (1.1), q is the potential vorticity, $\vec{v}$ is the horizontal velocity field, ω, is the relative vorticity, and ψ is the stream function. The horizontal space variables are given by $\vec{x} = (x, y)$ and t denotes time. The term βy is called the beta-plane effect from the Coriolis force and its significance will be explained later. The term $h = h(x, y)$ represents the bottom floor topography. The term $\mathcal{D}(\Delta)\psi$ represents various possible dissipation mechanisms. Finally, the term $\mathcal{F}(\vec{x}, t)$ accounts for additional external forcing. The fluid density is set to 1.

Before continuing, we would like to explain briefly, in physical terms and without going into any technical details, the origin of the barotropic quasi-geostrophic equations. The barotropic rotational equations, also called rotating shallow water equations (Pedlosky, 1987), admit two different modes of propagation, slow and fast. The slow mode of propagation corresponds to the motion of the bulk of the fluid by advection. This is the slow motion we see in the weather patterns in the atmosphere, evolving on a time scale of days. The fast mode corresponds to gravity waves, which evolve on a short time scale of the order of several minutes, but do not contribute to the bulk motion of the fluid. The barotropic quasi-geostrophic equations are the result of "filtering out" the fast gravity waves from the rotating barotropic equations. There is also a formal analogy between barotropic quasi-geostrophic equations and incompressible flows; in the theory of compressible fluid flows the incompressible limit is obtained by "filtering out" the "fast" acoustic waves and retaining only the "slow" vortical modes associated to convection by the fluid (Majda, 1984). Indeed, it was this analogy that originally inspired Charney (1949) when he first formulated the quasi-geostrophic equations and thus opened the modern era of numerical weather prediction (Charney, 1949; Charney, Fjörtoft, and von Neumann, 1950).

The full derivation of the rotating barotropic equations and the corresponding barotropic quasi-geostrophic equations is lengthy and will take us too far from our main objective, which is the study of the quasi-geostrophic equations. For a thorough treatment of the barotropic rotational equations the reader is referred to Pedlosky (1987). Formal as well as rigorous derivations of the barotropic quasi-geostrophic equations from the rotating shallow water equations can be found in Majda (2003), Embid and Majda (1996).

Rather than deriving the quasi-geostrophic equations, we would like to explain the physical meaning and significance of the different terms appearing in equation (1.1). For barotropic quasi-geostrophic flows, the potential vorticity q is made of three different contributions. The first term $\omega = \Delta\psi = \operatorname{curl}\vec{v}$ is the fluid vorticity and represents the local rate of rotation of the fluid. The second term βy is the beta-plane effect from the Coriolis force and its appearance will be explained later. The third term $h = h(x, y)$ represents the bottom topography, as given by the ocean floor or a mountain range.

The horizontal velocity field, $\vec{v}$, is determined by the orthogonal gradient of the stream function ψ, $\vec{v} = \nabla^{\perp}\psi$, where the orthogonal gradient of ψ is defined as

$$\nabla^{\perp}\psi = \begin{pmatrix} -\dfrac{\partial\psi}{\partial y} \\ \\ \dfrac{\partial\psi}{\partial x} \end{pmatrix}.$$

In particular, the velocity field $\vec{v}$ is incompressible because

$$\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = \nabla\cdot\nabla^{\perp}\psi = 0.$$

The reason ψ is called the stream function is because at any fixed instant in time the velocity field $\vec{v}$ is perpendicular to the gradient of ψ, i.e. $\vec{v}$ is tangent to the level curves of ψ. Therefore the level curves of ψ represent the streamlines of the fluid. In addition, there is another important interpretation of ψ. Physically ψ represents the (hydrostatic) pressure of the fluid. In this context, the equation $\vec{v} = \nabla^{\perp}\psi$ corresponds to the fact that the flow field is in *geostrophic balance*, and therefore the streamlines also happen to be the *isobars* of the flow. In particular, we conclude that for a steady solution of the quasi-geostrophic equations the fluid flows along the isobars. This is in marked contrast with the situation in non-rotating fluids, where typically the flow is from regions of high pressure to those of low pressure.

The importance of the potential vorticity q is in the fact that it completely determines the state of the flow. Indeed in the barotropic quasi-geostrophic equations, once we know the potential vorticity q, the second equation in equation (1.1) immediately yields the vorticity ω. Since $\omega = \Delta\psi$, we can determine the stream function ψ, and then introduce it into the third equation in equation (1.1), namely $\vec{v} = \nabla^{\perp}\psi$, to determine the advective velocity field.

Next we return to a brief discussion of the beta-plane effect (cf. Pedlosky, 1987). This effect is essentially the result of linearizing the Coriolis force when we consider the motion of the fluid in the tangent plane approximation. More specifically, although the earth is spherical, we assume that the spatial scale of motion is moderate enough so that the region occupied by the fluid can be approximated by a tangent plane (this is certainly the case for mesoscale flows, even for horizontal ranges of the order of 10^3 km). This is what is called the tangent plane approximation. The equations of motion in equation (1.1) are written in terms of horizontal Cartesian coordinates in the tangent plane. In this context, the spatial variable x corresponds to longitude (with positive direction towards the east) and the variable y to latitude (with positive direction towards the north).[1] In fact, throughout this book we often refer to flows pointing in the positive (negative) x-direction as eastward (westward). Since the tangent plane rotates with the earth it becomes a non-inertial frame, and the Coriolis force due to the earth's rotation becomes an important effect in geophysical flows. Moreover, because of the curvature of the earth, the contribution of the Coriolis force depends on the latitude at which the tangent plane is being considered; the Coriolis force will increase from zero at the equator to its maximum value at the poles. Since the tangent plane approximation assumes a moderate range in latitude and longitude, a Taylor expansion approximation of the Coriolis force is permissible; the linear term of this Taylor expansion yields the beta-plane effect βy considered in equation (1.1). For the actual details of the tangent plane approximation and the beta-plane effect, the reader is encouraged to consult Pedlosky (1987) or Gill (1982).

[1] For simplicity we will always assume that the tangent plane approximation is considered in the northern hemisphere

There are many choices of dissipation operator $\mathcal{D}(\Delta)$, ranging from Ekman drag to Newtonian viscosity or hyper-viscosity. We list some commonly used dissipation operators below for later convenience:

(i) Newtonian (eddy) viscosity

$$\mathcal{D}(\Delta)\psi = \nu\Delta^2\psi$$

This form of the diffusion is identical to the ordinary molecular friction in a Newtonian fluid. For geophysical flows, the value of the coefficient is often assumed to be many orders of magnitude larger than that for molecular viscosity, and represents, crudely, smaller-scale turbulence effects. This led to the name, eddy viscosity.

(ii) Ekman drag dissipation

$$\mathcal{D}(\Delta)\psi = -d\Delta\psi,$$

which is common to the large-scale pieces of the geophysical flow. This arises from boundary layer effects in rapidly rotating flows.

(iii) Hyper-viscosity dissipation

$$\mathcal{D}(\Delta)\psi = (-1)^j d_j \Delta^j \psi, \quad j = 3, 4, 5, \cdots$$

This form of the dissipation term is frequently utilized in the study and numerical simulation of geophysical flows, where its role is to introduce very little dissipation in the large scales of the flow but to strongly damp out the small scales. The validity of the use of such hyper-viscous mechanisms is still an open issue among geophysical fluid dynamicists.

(iv) Ekman drag dissipation + Hyper-viscosity

$$\mathcal{D}(\Delta)\psi = -d\Delta\psi + (-1)^j d_j \Delta^j \psi, \qquad d_j \geq 0, \quad d > 0, \quad j > 2.$$

This is a combination of the previous two dissipation mechanisms.

(v) Radiative damping

$$\mathcal{D}(\Delta)\psi = d\psi$$

This represents a crude model for radiative damping when models with stratification are involved. Radiative damping is an unusual dissipation operator since it damps the large scales more strongly than the small scales in contrast to the standard diffusion operators in (i) and (iii) above.

(vi) General dissipation operator

$$\mathcal{D}(\Delta)\psi = \sum_{j=0}^{l} (-1)^j d_j \Delta^j \psi,$$

which encompasses all other forms of dissipation mechanisms previously discussed.

For simplicity we will consider periodic boundary conditions for the flow in both the x and y variables, say with period 2π in both variables

$$\begin{aligned}\vec{v}(x+2\pi, y, t) &= \vec{v}(x, y, t),\\ \vec{v}(x, y+2\pi, t) &= \vec{v}(x, y, t),\end{aligned} \tag{1.2}$$

or in terms of the stream function ψ

$$\psi(x+2\pi, y, t) = \psi(x, y+2\pi, t) = \psi(x, y, t). \tag{1.3}$$

We may also impose the zero average condition

$$\int \psi(x, y, t)\, dxdy = 0, \tag{1.4}$$

since the stream function is always determined up to a constant, and we can choose the constant here so that the average is zero. The assumption of periodicity in both variables is not unreasonable (except near drastic topographical barriers, such as continents). It allows us to use Fourier series and separation of variables as a main mathematical tool (see page 10 for a Fourier series tool kit). Physically, periodicity allows us to avoid other issues such as the appearance of boundary layers or the generation of vorticity at the boundary. However, occasionally we will consider other boundary conditions besides the periodic one. In particular, we will study flows in channel domains or in a rectangular basin which can be treated through minor modification of periodic flows with special geometry.

It is worthwhile to point out that, in the special case where there are no beta-plane effects or bottom floor topography, i.e. $\beta = 0$, $h = 0$, then the potential vorticity q reduces to the vorticity ω, $q = \omega$, and if we assume Newtonian dissipation, then the barotropic quasi-geostrophic equations reduce to the classical Navier–Stokes equations for a two-dimensional flow, written in the vorticity-stream form (Majda and Bertozzi, 2001; Chorin and Marsden, 1993)

Two-dimensional classical fluid flow equations

$$\frac{D\omega}{Dt} = \nu\Delta\omega + \mathcal{F}(\vec{x}, t), \quad \omega = \Delta\psi, \quad \vec{v} = \nabla^{\perp}\psi \tag{1.5}$$

and in the case without dissipation we have the classical Euler equations with forcing

$$\frac{D\omega}{Dt} = \mathcal{F}(\vec{x}, t), \quad \omega = \Delta\psi, \quad \vec{v} = \nabla^{\perp}\psi. \tag{1.6}$$

One of our objectives of this book is to compare and contrast the barotropic quasi-geostrophic equations and the Navier–Stokes equations to better understand the role of the beta-plane effect and the topography on the behavior of geophysical flows.

Even though we have restricted ourselves to the study of the barotropic quasi-geostrophic equations, it is still not possible for us (and not our intention) to cover all the possible problems associated with these equations. Instead we will focus on various topics that we consider physically interesting, yet mathematically tractable. We are especially interested in geophysical fluid flow phenomena, influenced by the presence of the Coriolis force and topography, combined with the presence of various dissipative and external forcing mechanisms, and on their role in the emergence and persistence of large coherent structures, as observed in mesoscale flows. However, many of the ideas and techniques apply to more complex models for geophysical flows, such as the F-plane equations, two layer models, continuously stratified quasi-geostrophic flow. The final section of this chapter discusses all of these models briefly as well as the inter-relations among them and the basic barotropic model. Generalizations of some of the material in the course to these equations are straightforward, while other material involves subtle current research.

Here we include a list of some of the topics that we will study in subsequent chapters:

(i) Exact solutions showing interesting physics.
There are many interesting patterns in geophysical flows ranging from Rossby waves to jets. One of our tasks here is to present some special exact solutions. They will illustrate simple Rossby wave motion, Taylor vortices, shear flows, simple topographic effects, etc.

(ii) Conserved quantities.
Conserved quantities play an essential role in both the physical understanding and mathematical study of geophysical flows. In this book we will carefully study various conserved quantities. A set of important conserved quantities are summarized, with the geophysical effects and domain geometry effects distinguished. These conserved quantities will play a central role in the subsequent study of non-linear stability of geophysical flows and the statistical theories of large-scale coherent structures.

(iii) Response to large-scale forcing.
We will establish the stability of motion on the ground shell, provided that the forcing is of the largest scale and dissipation is present. This will provide us with an explicit example of stable large coherent structure in damped and driven environment.

(iv) Selective decay.
We will demonstrate various facets of selective decay, both numerically and mathematically. This is an interesting example of how the inverse cascade is observed in two-dimensional flows.

(v) Non-linear stability of certain steady geophysical flows.
Stability is directly related to the issue of whether a specific flow is observable. Here the non-linear stability for certain geophysical flows is discussed, utilizing the Arnold–Kruskal method.

(vi) Large- and small-scale interaction via topographic stress.
Interesting phenomena arise when we introduce topography. An effective new stress, called topographic stress, effectively mediates the energy exchange between the large- and small-scale flows. We will see below that this is an interesting source of explicit examples with chaotic dynamics.

(vii) Equilibrium statistics mechanics and statistical theories for large coherent structures.
Here we develop a self-contained treatment of equilibrium statistical mechanics for geophysical flows in an elementary fashion. We develop elementary models with statistical features of the atmosphere and ocean, and equilibrium statistical theories for large coherent structures. The main perspective in achieving this is through information theory, which is developed in the text in a self-contained fashion.
If we are interested in large coherent structures instead of the small-scale fine structures, equilibrium statistical theory provides a way to predict the large coherent structure without calculating the details of the solutions. Various approaches will be presented. These will include the classical statistical theory with two conserved quantities, theories that attempt to incorporate infinitely many conserved quantities, and the current statistical theory with a few judicious constraints. In addition, a special numerical laboratory is developed and utilized to compare these approaches quantitatively.

(viii) Crude dynamic modeling for geophysical flows.
We will develop novel ways in which the equilibrium statistical theories can represent complex flows with damping and driving. This will include crude closure algorithms for both the classical fluid flows and flows with topography. This study will rely heavily on numerical evidence, but will also be supported with mathematical analysis.

(ix) We will apply the ideas developed in (vii) and (viii) to successfully predict the Great Red Spot of Jupiter in a fashion which is completely self-consistent with the observational record from the Voyager and Galileo space missions.

(x) Barotropic quasi-geostrophic equations on the sphere.
Actual fluid flow in the atmosphere occurs on the sphere and there are important new effects. Here all the previous problems will be reconsidered on the sphere. Some peculiar phenomena arise due to the special spherical symmetry.

(xi) We will show how information theory can be used to quantify predictability for ensemble predictions in geophysical flows. The following issues will be addressed. How important is the mean compared with the variance? When is a prediction reliably bi-modal with two different scenarios?

1.2 Some special exact solutions

Next we introduce and describe several families of special exact solutions of the barotropic quasi-geostrophic equations, equation (1.1). These solutions will be given with increasing levels of complexity, as we add more physical effects into the quasi-geostrophic equations. We will start by considering special steady flows free from beta-plane, topographical, diffusive, and external forcing effects. Even in this simplified situation, we will find a rich family of simple flows with interesting flow topology, which include shear flows, array of eddies, and Taylor vortices. Then we continue by systematically adding beta-plane effects, dissipation, and special external forcing, known as generalized Kolmogorov forcing. In particular, in this situation we will find flows with a large-scale mean flow and Rossby waves. We will also study the effects from the bottom floor topography and how it modifies the vorticity of the flow. This will be followed by examples incorporating the combined effects of the beta-plane and the topography. Finally, we conclude this section with an example of interaction of a large-scale shear flow with beta-plane dynamics. These special exact solutions are invaluable. They help us to build intuition and insight by revealing explicitly the behavior of the flow under the different physical mechanisms. They also provide us with ideal examples to test numerical methods designed to solve the quasi-geostrophic equations, as well as further theories about these geophysical flows.

In general it is far from easy to find exact solutions of the barotropic quasi-geostrophic equations. The difficulty lies in the non-linear character of the equations, through the non-linear advection term

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \vec{v} \cdot \nabla q.$$

Since the velocity $\vec{v}$ is given by the perpendicular gradient of the stream function ψ, $\vec{v} = \nabla^{\perp}\psi$, we can rewrite this non-linear term as

$$\nabla^{\perp}\psi \cdot \nabla q = \det \begin{pmatrix} \nabla\psi \\ \nabla q \end{pmatrix} = \begin{vmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix} = J(\psi, q),$$

where $J(\psi, q)$ is the Jacobian determinant of q and ψ. Therefore the potential vorticity equation in equation (1.1) takes the form

$$\frac{\partial q}{\partial t} + J(\psi, q) = \mathcal{D}(\Delta)\psi + \mathcal{F}(\vec{x}, t), \tag{1.7}$$

or equivalently

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q = \mathcal{D}(\Delta)\psi + \mathcal{F}(\vec{x}, t),$$

where $q = \Delta\psi + \beta y + h$.

To eliminate the non-linearity we must require the vanishing of the Jacobian determinant $J(\psi, q)$, and this certainly happens if the potential vorticity q and the stream function ψ are functionally dependent, i.e. if $q = F(\psi)$ for some function F.

Although such an assumption makes the potential vorticity equation linear, it also makes the elliptic equation for the stream function ψ non-linear

$$F(\psi) = q = \omega + \beta y + h = \Delta\psi + \beta y + h.$$

Clearly, any q and ψ, functionally related by $q = F(\psi)$ automatically define a steady (time-independent) exact solution of the barotropic quasi-geostrophic equation without damping or external forcing. In later chapters, the reader will find many examples of solutions of this type.

Here we concentrate on finding special exact solutions with both forcing and dissipation with the stronger ansatz: we *assume* that q and ψ are linearly dependent

$$q = \mu\psi, \tag{1.8}$$

then the elliptic equation for ψ now also becomes linear.

Summarizing, *under the linear dependence assumption* $q = \mu\psi$, the solution of the barotropic quasi-geostrophic equations, equation (1.1), is given by the:

Reduced linear system for the stream function ψ

$$\mu\frac{\partial\psi}{\partial t} = \mathcal{D}(\Delta)\psi + \mathcal{F}(\vec{x}, t), \quad \mu\psi = \Delta\psi + \beta y + h(\vec{x}), \tag{1.9}$$

where the velocity field $\vec{v}$, the vorticity ω, and the potential vorticity q are then given in terms of the stream function ψ by

$$\vec{v} = \nabla^{\perp}\psi, \quad \omega = \Delta\psi\ , \quad q = \mu\psi. \tag{1.10}$$

Throughout this book, we study geophysical flows in idealized periodic geometry or on special domains, such as channels or the square, which are related to the periodic geometry though symmetry considerations. Unless noted otherwise, all domains are 2π-periodic in each direction. Since the domain is periodic and the equations for the stream function ψ in equation (1.7) are linear, it is a natural desire to use the Fourier series as the main tool to study them. The only potential impediment may come from the beta-plane term βy, which is not a periodic function. However, we will solve this problem later with the introduction of a suitable large-scale mean flow for the velocity field.

Next we summarize some important basic properties of the Fourier series, which we will use throughout the book, and then discuss some particular solutions of the linear equations in equation (1.7).

Fourier series tool kit

Here we recall a few basic properties of the Fourier series that are used in this book.