1

BASIC HYPERGEOMETRIC SERIES

1.1 Introduction

Our main objective in this chapter is to present the definitions and notations for hypergeometric and basic hypergeometric series, and to derive the elementary formulas that form the basis for most of the summation, transformation and expansion formulas, basic integrals, and applications to orthogonal polynomials and to other fields that follow in the subsequent chapters. We begin by defining Gauss' ${}_2F_1$ hypergeometric series, the ${}_rF_s$ (generalized) hypergeometric series, and pointing out some of their most important special cases. Next we define Heine's ${}_2\phi_1$ basic hypergeometric series which contains an additional parameter q, called the base, and then give the definition and notations for ${}_r\phi_s$ basic hypergeometric series. Basic hypergeometric series are called q-analogues (basic analogues or q-extensions) of hypergeometric series because an ${}_rF_s$ series can be obtained as the $q \to 1$ limit case of an ${}_r\phi_s$ series.

Since the binomial theorem is at the foundation of most of the summation formulas for hypergeometric series, we then derive a q-analogue of it, called the q-binomial theorem, and use it to derive Heine's q-analogues of Euler's transformation formulas, Jacobi's triple product identity, and summation formulas that are q-analogues of those for hypergeometric series due to Chu and Vandermonde, Gauss, Kummer, Pfaff and Saalschütz, and to Karlsson and Minton. We also introduce q-analogues of the exponential, gamma and beta functions, as well as the concept of a q-integral that allows us to give a q-analogue of Euler's integral representation of a hypergeometric function. Many additional formulas and q-analogues are given in the exercises at the end of the chapter.

1.2 Hypergeometric and basic hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss [1813]) in which he considered the infinite series

$$1+\frac{ab}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}z^3+\cdots \quad (1.2.1)$$

as a function of a, b, c, z, where it is assumed that $c \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $\mathrm{Re}\ (c-a-b) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula (see (1.2.11) below) for the sum of this series when $z = 1$ and $\mathrm{Re}\ (c-a-b) > 0$.

Although Gauss used the notation $F(a,b,c,z)$ for his series, it is now customary to use $F(a,b;c;z)$ or either of the notations

$$ {}_2F_1(a,\ b;\ c;\ z), \qquad {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix}; z\right] $$

for this series (and for its sum when it converges), because these notations separate the numerator parameters a,b from the denominator parameter c and the variable z. In view of Gauss' paper, his series is frequently called *Gauss' series*. However, since the special case $a=1, b=c$ yields the geometric series

$$ 1+z+z^2+z^3+\cdots, $$

Gauss' series is also called the *(ordinary) hypergeometric series* or the *Gauss hypergeometric series*.

Some important functions which can be expressed by means of Gauss' series are

$$ \begin{aligned} (1+z)^a &= F(-a,b;b;-z), \\ \log(1+z) &= zF(1,1;2;-z), \\ \sin^{-1} z &= zF(1/2,1/2;3/2;z^2), \\ \tan^{-1} z &= zF(1/2,1;3/2;-z^2), \\ e^z &= \lim_{a\to\infty} F(a,b;b;z/a), \end{aligned} \tag{1.2.2} $$

where $|z|<1$ in the first four formulas. Also expressible by means of Gauss' series are the classical orthogonal polynomials, such as the *Tchebichef polynomials of the first and second kinds*

$$ T_n(x) = F(-n,n;1/2;(1-x)/2), \tag{1.2.3} $$

$$ U_n(x) = (n+1)F(-n,n+2;3/2;(1-x)/2), \tag{1.2.4} $$

the *Legendre polynomials*

$$ P_n(x) = F(-n,n+1;1;(1-x)/2), \tag{1.2.5} $$

the *Gegenbauer (ultraspherical) polynomials*

$$ C_n^\lambda(x) = \frac{(2\lambda)_n}{n!}\, F(-n,n+2\lambda;\lambda+1/2;(1-x)/2), \tag{1.2.6} $$

and the more general *Jacobi polynomials*

$$ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, F(-n,n+\alpha+\beta+1;\ \alpha+1;(1-x)/2), \tag{1.2.7} $$

where $n=0,1,\ldots,$ and $(a)_n$ denotes the *shifted factorial* defined by

$$ (a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n=1,2,\ldots . \tag{1.2.8} $$

Before Gauss, Chu [1303] (see Needham [1959, p. 138], Takács [1973] and Askey [1975, p. 59]) and Vandermonde [1772] had proved the summation formula

$$ F(-n,b;c;1) = \frac{(c-b)_n}{(c)_n}, \quad n=0,1,\ldots, \tag{1.2.9} $$

which is now called *Vandermonde's formula* or the *Chu–Vandermonde formula*, and Euler [1748] had derived several results for hypergeometric series, including his transformation formula

$$F(a,b;c;z) = (1-z)^{c-a-b}\, F(c-a, c-b; c; z), \quad |z| < 1. \tag{1.2.10}$$

Formula (1.2.9) is the terminating case $a = -n$ of the summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, \tag{1.2.11}$$

which Gauss proved in his paper.

Thirty-three years after Gauss' paper, Heine [1846, 1847, 1878] introduced the series

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})} z^2 + \cdots, \tag{1.2.12}$$

where it is assumed that $q \neq 1$, $c \neq 0, -1, -2, \ldots$ and the principal value of each power of q is taken. This series converges absolutely for $|z| < 1$ when $|q| < 1$ and it tends (at least termwise) to Gauss' series as $q \to 1$, because

$$\lim_{q \to 1} \frac{1-q^a}{1-q} = a. \tag{1.2.13}$$

The series in (1.2.12) is usually called *Heine's series* or, in view of the base q, the *basic hypergeometric series* or *q-hypergeometric series*.

Analogous to Gauss' notation, Heine used the notation $\phi(a,b,c,q,z)$ for his series. However, since one would like to also be able to consider the case when q to the power a, b, or c is replaced by zero, it is now customary to define the *basic hypergeometric series* by

$$\begin{aligned} \phi(a,b;c;q,z) \equiv {}_2\phi_1(a,b;c;q,z) &\equiv {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \end{aligned} \tag{1.2.14}$$

where

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots, \end{cases} \tag{1.2.15}$$

is the *q-shifted factorial* and it is assumed that $c \neq q^{-m}$ for $m = 0, 1, \ldots$. Some other notations that have been used in the literature for the product $(a;q)_n$ are $(a)_{q,n}, [a]_n$, and even $(a)_n$ when (1.2.8) is not used and the base is not displayed.

Another generalization of Gauss' series is the *(generalized) hypergeometric series* with r numerator parameters $a_1, \ldots, a_r$ and s denominator parameters $b_1, \ldots, b_s$ defined by

$$\begin{aligned} {}_rF_s(a_1, a_2, \ldots, a_r;\ b_1, \ldots, b_s; z) &\equiv {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{n!(b_1)_n \cdots (b_s)_n} z^n. \end{aligned} \tag{1.2.16}$$

Some well-known special cases are the *exponential function*

$$e^z = {}_0F_0(—; —; z), \tag{1.2.17}$$

the *trigonometric functions*

$$\begin{aligned} \sin z &= z\ {}_0F_1(—; 3/2; -z^2/4), \\ \cos z &= {}_0F_1(—; 1/2; -z^2/4), \end{aligned} \tag{1.2.18}$$

the *Bessel function*

$$J_\alpha(z) = (z/2)^\alpha\ {}_0F_1(—; \alpha+1; -z^2/4)/\Gamma(\alpha+1), \tag{1.2.19}$$

where a dash is used to indicate the absence of either numerator (when $r = 0$) or denominator (when $s = 0$) parameters. Some other well-known special cases are the *Hermite polynomials*

$$H_n(x) = (2x)^n\ {}_2F_0(-n/2, (1-n)/2; —; -x^{-2}), \tag{1.2.20}$$

and the *Laguerre polynomials*

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n; \alpha+1; x). \tag{1.2.21}$$

Generalizing Heine's series, we shall define an ${}_r\phi_s$ *basic hypergeometric series* by

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) &\equiv {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] \\ = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} &\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{1.2.22}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $r > s+1$.

In (1.2.16) and (1.2.22) it is assumed that the parameters $b_1, \ldots, b_s$ are such that the denominator factors in the terms of the series are never zero. Since

$$(-m)_n = \left(q^{-m}; q\right)_n = 0, \quad n = m+1, m+2, \ldots, \tag{1.2.23}$$

an ${}_rF_s$ series terminates if one of its numerator parameters is zero or a negative integer, and an ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-m} with $m = 0, 1, 2, \ldots$, and $q \neq 0$. Basic analogues of the classical orthogonal polynomials will be considered in Chapter 7 as well as in the exercises at the ends of the chapters.

Unless stated otherwise, when dealing with nonterminating basic hypergeometric series we shall assume that $|q| < 1$ and that the parameters and variables are such that the series converges absolutely. Note that if $|q| > 1$, then we can perform an inversion with respect to the base by setting $p = q^{-1}$ and using the identity

$$(a; q)_n = (a^{-1}; p)_n (-a)^n p^{-\binom{n}{2}} \tag{1.2.24}$$

to convert the series (1.2.22) to a similar series in base p with $|p| < 1$ (see Ex. 1.4(i)). The inverted series will have a finite radius of convergence if the original series does.

Observe that if we denote the terms of the series (1.2.16) and (1.2.22) which contain z^n by u_n and v_n, respectively, then

$$\frac{u_{n+1}}{u_n} = \frac{(a_1+n)(a_2+n)\cdots(a_r+n)}{(1+n)(b_1+n)\cdots(b_s+n)}\, z \tag{1.2.25}$$

is a rational function of n, and

$$\frac{v_{n+1}}{v_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}\,\left(-q^n\right)^{1+s-r} z \tag{1.2.26}$$

is a rational function of q^n. Conversely, if $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are power series with $u_0 = v_0 = 1$ such that u_{n+1}/u_n is a rational function of n and v_{n+1}/v_n is a rational function of q^n, then these series are of the forms (1.2.16) and (1.2.22), respectively.

By the ratio test, the ${}_rF_s$ series converges absolutely for all z if $r \le s$, and for $|z| < 1$ if $r = s+1$. By an extension of the ratio test (Bromwich [1959, p. 241]), it converges absolutely for $|z| = 1$ if $r = s+1$ and $\mathrm{Re}\,[b_1 + \cdots + b_s - (a_1 + \cdots + a_r)] > 0$. If $r > s+1$ and $z \ne 0$ or $r = s+1$ and $|z| > 1$, then this series diverges, unless it terminates.

If $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$. It diverges for $z \ne 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|z| > |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$, unless it terminates. As is customary, the ${}_rF_s$ and ${}_r\phi_s$ notations are also used for the sums of these series inside the circle of convergence and for their analytic continuations (called *hypergeometric functions* and *basic hypergeometric functions*, respectively) outside the circle of convergence.

Observe that the series (1.2.22) has the property that if we replace z by z/a_r and let $a_r \to \infty$, then the resulting series is again of the form (1.2.22) with r replaced by $r-1$. Because this is not the case for the ${}_r\phi_s$ series defined without the factors $\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$ in the books of Bailey [1935] and Slater [1966] and we wish to be able to handle such limit cases, we have chosen to use the series defined in (1.2.22). There is no loss in generality since the Bailey and Slater series can be obtained from the $r = s+1$ case of (1.2.22) by choosing s sufficiently large and setting some of the parameters equal to zero.

An ${}_{r+1}F_r$ series is called *k-balanced* if $b_1 + b_2 + \cdots + b_r = k + a_1 + a_2 + \cdots + a_{r+1}$ and $z = 1$; a 1-balanced series is called *balanced* (or *Saalschützian*). Analogously, an ${}_{r+1}\phi_r$ series is called *k-balanced* if $b_1b_2\cdots b_r = q^k a_1a_2\cdots a_{r+1}$ and $z = q$, and a 1-balanced series is called *balanced* (or *Saalschützian*). We will first encounter balanced series in §1.7, where we derive a summation formula for such a series.

For negative subscripts, the *shifted factorial* and the *q-shifted factorials* are defined by

$$(a)_{-n} = \frac{1}{(a-1)(a-2)\cdots(a-n)} = \frac{1}{(a-n)_n} = \frac{(-1)^n}{(1-a)_n}, \tag{1.2.27}$$

$$(a;q)_{-n} = \frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \tag{1.2.28}$$

where $n = 0, 1, \ldots$. We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{1.2.29}$$

for $|q| < 1$. Since the infinite product in (1.2.29) diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. The following easily verified identities will be frequently used in this book:

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1.2.30}$$

$$(a^{-1}q^{1-n};q)_n = (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}, \tag{1.2.31}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}\,(-qa^{-1})^k q^{\binom{k}{2}-nk}, \tag{1.2.32}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \tag{1.2.33}$$

$$(aq^n;q)_k = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}, \tag{1.2.34}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}, \tag{1.2.35}$$

$$(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}, \tag{1.2.36}$$

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \tag{1.2.37}$$

$$(aq^{-n};q)_k = \frac{(a;q)_k(qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n}q^{-nk}, \tag{1.2.38}$$

$$(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n, \tag{1.2.39}$$

$$(a^2;q^2)_n = (a;q)_n(-a;q)_n, \tag{1.2.40}$$

where n and k are integers. A more complete list of useful identities is given in Appendix I at the end of the book.

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$(a_1, a_2, \ldots, a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.2.41}$$

$$(a_1, a_2, \ldots, a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty. \tag{1.2.42}$$

The ratio $(1-q^a)/(1-q)$ considered in (1.2.13) is called a *q-number* (or *basic number*) and it is denoted by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1. \tag{1.2.43}$$

It is also called a q-analogue, q-deformation, q-extension, or a q-generalization of the complex number a. In terms of q-numbers the *q-number factorial* $[n]_q!$ is defined for a nonnegative integer n by

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \tag{1.2.44}$$

and the corresponding *q-number shifted factorial* is defined by

$$[a]_{q;n} = \prod_{k=0}^{n-1} [a+k]_q. \tag{1.2.45}$$

Clearly,

$$\lim_{q\to 1} [n]_q! = n!, \quad \lim_{q\to 1} [a]_q = a, \tag{1.2.46}$$

and

$$[a]_{q;n} = (1-q)^{-n}(q^a;q)_n, \quad \lim_{q\to 1} [a]_{q;n} = (a)_n. \tag{1.2.47}$$

Corresponding to (1.2.41) we can use the compact notation

$$[a_1, a_2, \ldots, a_m]_{q;n} = [a_1]_{q;n}[a_2]_{q;n}\cdots[a_m]_{q;n}. \tag{1.2.48}$$

Since

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r]_{q;n}}{[n]_q![b_1, \ldots, b_s]_{q;n}} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \\ &= {}_r\phi_s\left(q^{a_1}, q^{a_2}, \ldots, q^{a_r}; q^{b_1}, \ldots, q^{b_s}; q, z(1-q)^{1+s-r}\right), \end{aligned} \tag{1.2.49}$$

anyone working with q-numbers and the *q-number hypergeometric series* on the left-hand side of (1.2.49) can use the formulas for ${}_r\phi_s$ series in this book that have no zero parameters by replacing the parameters by q^{th} powers and applying (1.2.49).

As in Frenkel and Turaev [1995] one can define a *trigonometric number* $[a;\sigma]$ by

$$[a;\sigma] = \frac{\sin(\pi\sigma a)}{\sin(\pi\sigma)} \tag{1.2.50}$$

for noninteger values of σ and view $[a;\sigma]$ as a *trigonometric deformation* of a since $\lim_{\sigma\to 0}[a;\sigma] = a$. The corresponding ${}_rt_s$ *trigonometric hypergeometric series* can be defined by

$$\begin{aligned} &{}_rt_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; \sigma, z) \\ &= \sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r; \sigma]_n}{[n;\sigma]![b_1, \ldots, b_s; \sigma]_n} \left[(-1)^n e^{\pi i \sigma \binom{n}{2}}\right]^{1+s-r} z^n, \end{aligned} \tag{1.2.51}$$

where

$$[n;\sigma]! = \prod_{k=1}^{n}[k;\sigma], \quad [a;\sigma]_n = \prod_{k=0}^{n-1}[a+k;\sigma], \tag{1.2.52}$$

and

$$[a_1,a_2,\ldots,a_m;\sigma]_n = [a_1;\sigma]_n[a_2;\sigma]_n\cdots[a_m;\sigma]_n. \tag{1.2.53}$$

From

$$[a;\sigma] = \frac{e^{\pi i\sigma a}-e^{-\pi i\sigma a}}{e^{\pi i\sigma}-e^{-\pi i\sigma}} = \frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}} = \frac{1-q^a}{1-q}q^{(1-a)/2}, \tag{1.2.54}$$

where $q = e^{2\pi i\sigma}$, it follows that

$$[a;\sigma]_n = \frac{(q^a;q)_n}{(1-q)^n}q^{n(1-a)/2-n(n-1)/4}, \tag{1.2.55}$$

and hence

$$\begin{aligned}&{}_rt_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;\sigma,z)\\&= {}_r\phi_s(q^{a_1},q^{a_2},\ldots,q^{a_r};q^{b_1},\ldots,q^{b_s};q,cz)\end{aligned} \tag{1.2.56}$$

with

$$c = (1-q)^{1+s-r}q^{r/2-s/2+(b_1+\cdots+b_s)/2-(a_1+\cdots+a_r)/2}, \tag{1.2.57}$$

which shows that the ${}_rt_s$ series is equivalent to the ${}_r\phi_s$ series in (1.2.49).

Elliptic numbers $[a;\sigma,\tau]$, which are a one-parameter generalization (deformation) of trigonometric numbers, are considered in §1.6, and the corresponding elliptic (and theta) hypergeometric series and their summation and transformation formulas are considered in Chapter 11.

We close this section with two identities involving ordinary binomial coefficients, which are particularly useful in handling some powers of q that arise in the derivations of many formulas containing q-series:

$$\binom{n+k}{2} = \binom{n}{2}+\binom{k}{2}+kn\,, \tag{1.2.58}$$

$$\binom{n-k}{2} = \binom{n}{2}+\binom{k}{2}+k-kn\,. \tag{1.2.59}$$

1.3 The q-binomial theorem

One of the most important summation formulas for hypergeometric series is given by the *binomial theorem*:

$${}_2F_1(a,c;c;z) = {}_1F_0(a;—;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n = (1-z)^{-a}, \tag{1.3.1}$$

where $|z|<1$. We shall show that this formula has the following q-analogue

$${}_1\phi_0(a;—;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1,\ |q|<1, \tag{1.3.2}$$

which was derived by Cauchy [1843], Heine [1847] and by other mathematicians. See Askey [1980a], which also cites the books by Rothe [1811] and Schweins [1820], and the remark on p. 491 of Andrews, Askey, and Roy [1999] concerning the terminating form of the q-binomial theorem in Rothe [1811].

Heine's proof of (1.3.2), which can also be found in the books Heine [1878], Bailey [1935, p. 66] and Slater [1966, p. 92], is better understood if one first follows Askey's [1980a] approach of evaluating the sum of the binomial series in (1.3.1), and then carries out the analogous steps for the series in (1.3.2).

Let us set

$$f_a(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \tag{1.3.3}$$

Since this series is uniformly convergent in $|z| \leq \epsilon$ when $0 < \epsilon < 1$, we may differentiate it termwise to get

$$\begin{aligned} f_a'(z) &= \sum_{n=1}^{\infty} \frac{n(a)_n}{n!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{n!} z^n = a f_{a+1}(z). \end{aligned} \tag{1.3.4}$$

Also

$$\begin{aligned} f_a(z) - f_{a+1}(z) &= \sum_{n=1}^{\infty} \frac{(a)_n - (a+1)_n}{n!} z^n \\ = \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}}{n!} &[a - (a+n)]\, z^n = - \sum_{n=1}^{\infty} \frac{n(a+1)_{n-1}}{n!} z^n \\ = - \sum_{n=0}^{\infty} \frac{(a+1)_n}{n!} z^{n+1} &= -z f_{a+1}(z). \end{aligned} \tag{1.3.5}$$

Eliminating $f_{a+1}(z)$ from (1.3.4) and (1.3.5), we obtain the first order differential equation

$$f_a'(z) = \frac{a}{1-z} f_a(z), \tag{1.3.6}$$

subject to the initial condition $f_a(0) = 1$, which follows from the definition (1.3.3) of $f_a(z)$. Solving (1.3.6) under this condition immediately gives that $f_a(z) = (1-z)^{-a}$ for $|z| < 1$.

Analogously, let us now set

$$h_a(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \quad |z| < 1,\ |q| < 1. \tag{1.3.7}$$

Clearly, $h_{q^a}(z) \to f_a(z)$ as $q \to 1$. Since $h_{aq}(z)$ is a q-analogue of $f_{a+1}(z)$, we first compute the difference

$$h_a(z) - h_{aq}(z) = \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n$$

$$= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} [1-a-(1-aq^n)]\, z^n$$
$$= -a \sum_{n=1}^{\infty} \frac{(1-q^n)(aq;q)_{n-1}}{(q;q)_n} z^n$$
$$= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_{n-1}} z^n = -azh_{aq}(z), \tag{1.3.8}$$

giving an analogue of (1.3.5). Observing that

$$f'(z) = \lim_{q\to 1} \frac{f(z)-f(qz)}{(1-q)z} \tag{1.3.9}$$

for a differentiable function f, we next compute the difference

$$h_a(z) - h_a(qz) = \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n}(z^n - q^n z^n)$$
$$= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_{n-1}} z^n = \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}}{(q;q)_n} z^{n+1}$$
$$= (1-a)zh_{aq}(z). \tag{1.3.10}$$

Eliminating $h_{aq}(z)$ from (1.3.8) and (1.3.10) gives

$$h_a(z) = \frac{1-az}{1-z} h_a(qz). \tag{1.3.11}$$

Iterating this relation $n-1$ times and then letting $n \to \infty$ we obtain

$$h_a(z) = \frac{(az;q)_n}{(z;q)_n} h_a(q^n z)$$
$$= \frac{(az;q)_\infty}{(z;q)_\infty} h_a(0) = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{1.3.12}$$

since $q^n \to 0$ as $n \to \infty$ and $h_a(0) = 1$ by (1.3.7), which completes the proof of (1.3.2).

One consequence of (1.3.2) is the product formula

$${}_1\phi_0(a;—;q,z)\; {}_1\phi_0(b;—;q,az) = {}_1\phi_0(ab;—;q,z), \tag{1.3.13}$$

which is a q-analogue of $(1-z)^{-a}(1-z)^{-b} = (1-z)^{-a-b}$.

In the special case $a = q^{-n}, n = 0, 1, 2, \ldots,$ (1.3.2) gives

$${}_1\phi_0(q^{-n};—;q,z) = (zq^{-n};q)_n = (-z)^n q^{-n(n+1)/2}(q/z;q)_n, \tag{1.3.14}$$

where, by analytic continuation, z can be any complex number. From now on, unless stated othewise, whenever $q^{-j}, q^{-k}, q^{-m}, q^{-n}$ appear as numerator parameters in basic series it will be assumed that j, k, m, n, respectively, are nonnegative integers.

If we set $a = 0$ in (1.3.2), we get

$${}_1\phi_0(0;—;q,z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},\ |z| < 1, \tag{1.3.15}$$