

Index

Abbreviations used in this index: *distn* distribution; *eqn* equation; *fn* function; *mgf* moment generating function; *pgf* probability generating function; *pr* process; *rv* random variable; *rw* random walk; *thm* theorem

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Index of Notation

$B(n, p)$	binomial distribution
$\text{cov}(X, Y)$	covariance
$f(x), f_X(x)$	mass or density function
$f_{X Y}(x y)$	conditional mass or density
$f_{X,Y}(x, y)$	joint mass or density
i	$\sqrt{-1}$
i, j, k, m, n	indices
$\max(x, y) = x \vee y$	maximum of x and y
$\min(x, y) = x \wedge y$	minimum of x and y
$p, p_i, p_{ij}, p(t), p_{ij}(t)$	probabilities
x^+	$\max(x, 0)$
$\text{var}(X)$	variance
$ A $	cardinality or size of A
A^c	complement of A
$B(a, b)$	beta function
$F(x), F_X(x)$	distribution function
$F_{X Y}(x y)$	conditional distribution
$F_{X,Y}(x, y)$	joint distribution
$G(s), G_X(s)$	generating function
I_A	indicator of the event A
J	Jacobian
$M_X(t)$	moment generating function
H, T	head, tail
$N(\mu, \sigma^2)$	normal distribution
$N(t)$	Poisson process
$X, Y, Z, X(\omega)$	random variables
$\mathbf{X}, \mathbf{Y}, \mathbf{W}$	random vectors
$W(t)$	Wiener process
\mathcal{F}	σ -field or event space
η	probability of extinction
$\chi^2(\cdot)$	chi-squared distribution
$\phi(x)$	standard normal density
$\Phi(x)$	standard normal distribution
\mathbf{x}	vector

\mathbf{x}^T	vector transpose
A	matrix
A^T	matrix transpose
$\mu^{(k)}$	fractional moment
μ	mean
μ_i	mean recurrence time
π, π_i	stationary distribution
σ^2	variance
$\rho(X, Y)$	correlation
$\Gamma(t)$	gamma function
Ω	sample space
$\mathbf{E}X$	expectation of X
$\mathbf{E}(X Y)$	conditional expectation
$\mathbf{P}(A)$	probability of A
$\mathbf{P}(A B)$	conditional probability
ϕ	empty set, impossible event
$O(\cdot), o(\cdot)$	order notation

ENVOY

Probability begins and ends with probability.

John Maynard Keynes.