Nonuniform Hyperbolicity

Dynamics of Systems with Nonzero Lyapunov Exponents

Designed to work as a reference and as a supplement to an advanced course on dynamical systems, this book presents a self-contained and comprehensive account of modern smooth ergodic theory. Among other things, this provides a rigorous mathematical foundation for the phenomenon known as deterministic chaos – the appearance of chaotic motions in pure deterministic dynamical systems. A sufficiently complete description of topological and ergodic properties of systems exhibiting deterministic chaos can be deduced from relatively weak requirements on their local behavior known as nonuniform hyperbolicity conditions.

Nonuniform hyperbolicity theory is an important part of the general theory of dynamical systems. Its core is the study of dynamical systems with nonzero Lyapunov exponents both conservative and dissipative, in addition to cocycles and group actions. The results of this theory are widely used in geometry (e.g., geodesic flows and Teichmüller flows), in rigidity theory, in the study of some partial differential equations (e.g., the Schrödinger equation), and in the theory of billiards, as well as in applications to physics, biology, engineering, and other fields.

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To Clàudia

and

To Natasha, Irishka, and Lenochka

for their patience, encouragement, and inspiration

Contents

Preface

Writing this book was a long-term project that has taken several years, and at the early stages Anatole Katok's participation was crucial. He provided us with the text of his unpublished notes with Leonardo Mendoza that served as the basis for the first draft of Chapters 1–5 and parts of Chapters 6–7. He also fully participated in designing the content of the book in its present form.

In this book we present a self-contained and sufficiently complete description of the modern nonuniform hyperbolicity theory, that is, the theory of dynamical systems whose Lyapunov exponents are not zero. The reader will find all the core results of the theory as well as a good account of its recent developments.

The nonuniform hyperbolicity theory is rich in wonderful ideas and sophisticated techniques, which are widely used in many areas of dynamical systems as well as other areas of mathematics and beyond. The nonuniform hyperbolicity theory is very popular and finds a lot of applications outside mathematics – in physics, biology, engineering, and so on.

Despite (or should we say because of) a tremendous amount of research on the subject, there have been relatively few attempts to summarize and unify the results of the theory in a single manuscript or a survey (see the books [110, 139, 179], the surveys [18, 173, 175], and the lectures [19, 27]). This book is meant to cover this gap. It can be used as a reference book for the theory or as a supporting material for an advanced course on dynamical systems. During the long course of working on this book, we first produced its baby version [20] in which we described the core results of the theory and some principal examples, and then we wrote the survey [21] in which we presented the contemporary status of the theory.

Since the beginning of the 1970s, the nonuniform hyperbolicity theory has emerged as an independent discipline lying in the heart of the modern theory of dynamical systems. It studies both conservative (volume preserving) and dissipative systems, deterministic as well as random dynamical systems, and discrete and continuous-time systems, in addition to cocycles and group actions. The results of this theory have found their way into geometry (e.g., in the study of geodesic flows and Teichmüller flows), into rigidity theory, into the study of some partial differential equations (e.g., the Schrödinger equation and some reaction-diffusion equations), and into the theory of "chaotic" billiards.

Writing a book of such a scope can be deemed a daunting task and we therefore had to select topics so that personal taste, clearly biased toward our own interests, entered in our choices. As a result, some interesting topics are barely mentioned or not covered at all. In particular, we do not consider random dynamical systems, referring the reader to the books [9, 112, 129] and the survey [113], nor dynamical systems, with singularities (see the book [110]), in particular, leaving aside the rich theory of chaotic billiards. We restrict ourselves to the case of invertible dynamical systems and thus the theory of nonuniformly expanding maps is not discussed here (see the survey [132]) nor do we include one-dimensional chaotic maps (e.g., the logistic family, see [94]). We touch upon some recent results on Hénon-like attractors related to the study of Sinai–Ruelle–Bowen measures but we do not go deep into the theory of these attractors (see the survey [133]). We mention some results on hyperbolic group actions and refer the reader to [71] for a more complete account.

All the principal results of the nonuniform hyperbolicity theory are presented in the book with complete proofs, although some other results are included, without proofs, for the sake of completeness.

Most chapters of the book end with notes in which the reader can find some remarks of historical and bibliographical nature, comments on some related results, and references for further reading. In no way are these notes meant to present a significant account of the history of the subject or a sufficiently complete list of references.

Acknowledgments

While working on the book, several people helped us in various ways and it is our great pleasure to acknowledge their contributions.

As we mentioned above, it is impossible to overestimate the contribution of Anatole Katok, whom we heartily thank for his constant support and guidance.

We also thank Marlies Gerber for providing us with the text of Section 6.4.2 and Theorem 9.2.8 and for some useful remarks.

It is our pleasure to thank Anton Zorich for useful comments on Section 12.5.

We are grateful to Claudia Valls, who carefully read several chapters of the book and made many remarks that helped us clarify the presentation.

When the first draft of the book was ready, we sent it to several experts in the field, asking for their opinions and comments. We are grateful to Dima Dolgopyat, François Ledrappier, Mark Pollicott, and Federico Rodríguez Hertz for innumerable constructive suggestions that helped us improve the content and presentation of some results of the book as well as extend and enhance the bibliography.

Our special thanks go to Boris Hasselblatt, who thoroughly examined the draft and pointed out many places in the book that needed additional work. Reflecting on his comments, we introduced many changes improving the style and exposition of the material; we also added more informal discussions, hopefully making the book more reader friendly.

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