Introduction

The aim of this book is to give an introduction to two-dimensional boundary conformal field theory (CFT) and an overview of its various applications to D-branes in string theory.

The study of two-dimensional scale invariant quantum field theories (QFTs) has a long history. Applications to important problems in various branches of physics are so numerous that CFT has established its position as one of the leading techniques in modern theoretical physics. Its first significant success was the exact computation of critical exponents for second-order phase transitions in two-dimensional statistical systems, such as the critical Ising model and many extensions thereof. Surface critical phenomena are still among the key applications, but many less obvious ones have joined.

The applications most relevant for us are those in string theory. To this date, string theory clearly offers the most promising candidate for a fundamental theory of quantum gravity and an intriguing approach to the unification of all four known interactions. Strings may be considered as one-dimensional quantum systems or, equivalently, two-dimensional statistical systems. Among all such systems, CFTs are singled out since they solve the string theoretic equations of motion.

Let us also mention that two-dimensional CFTs offer a very fruitful laboratory for QFT. Many models of CFT can be solved non-perturbatively, mostly aided by their infinite-dimensional symmetries. This offers unique insights into the very nature of QFT, the importance of non-perturbative effects and dualities between theories with seemingly different field content. In addition, most CFTs also possess perturbations which may break scale invariance but preserve integrability, so that analytic expressions for many quantities become available.

The analysis of boundary conditions is a natural problem in physics. All realistic two-dimensional statistical systems possess boundaries and hence their full theoretical understanding clearly requires a good control of boundary conditions. For two-dimensional CFT, the study of boundaries was pioneered by Cardy in a series of papers, in particular [98, 100]. Once again, the presence of powerful infinite-dimensional symmetries has led to many exact results on boundary critical exponents and correlation functions.

Boundary CFTs are in fact more directly applicable to "real" physical situations than their relatives on closed surfaces. Many processes in three space dimensions are dominated by scattering in the s-wave channel, where the relevant quantities depend on time and a radial coordinate. Therefore, QFTs on the half-plane appear naturally. Quantum impurity scattering (the Kondo effect) is the most famous example.

In string theory, boundaries enter through the description of open strings. But it was not until the mid nineties and the discovery of branes that the whole relevance of the plethora of boundary conditions for string theory was fully realised. At low spacetime energy, p-branes appear as supergravity solutions, which describe stable objects whose mass and charge is distributed along $(p+1)$-dimensional hypersurfaces in spacetime. Beyond the low-energy regime, supergravity needs to be replaced by string theory, and so a natural question to ask was how to describe branes in string theory. For a large class of branes, those that became known as D-branes, the answer was given by Polchinski in [375]: D-branes are objects on which open strings can end. The "D" in D-branes stands for Dirichlet boundary conditions, which force the open-string endpoints to stay within the brane worldvolume. String theory contains many different branes, which are characterised by their dimension and additional data to be described extensively below. All these data must be encoded in the choice of boundary conditions.

The importance of D-branes for our understanding of string theory, and perhaps even for the development of modern theoretical physics, can hardly be overestimated. After about 25 years of perturbative calculations, branes have made some non-perturbative features of string theory accessible. This "string revolution" has led to the conviction that string theories which previously were considered as independent are in fact merely different realisations of a single underlying theory, M-theory, see [437, 453] but also the earlier pre-D-brane proposals in [173]. The discussion of non-perturbative aspects is intimately related to the observation that string theory is more than a theory of one-dimensional objects travelling in a (quantised) spacetime, and that its consistent formulation requires the inclusion of higher-dimensional extended objects as well, namely membranes or D-branes.

As a "spin-off", D-branes have triggered much progress in the understanding of dualities in (supersymmetric) gauge theories in various dimensions, see e.g. [238, 329, 418] for some early results, and they continue to do so. Arguably the most profound development originating from brane physics, one that has influenced thousands of papers on many aspects of gauge theories and string theory in various dimensions, is Maldacena's intriguing anti-de Sitter (AdS)/CFT correspondence between (conformal) gauge theories and string theory in (asymptotically) AdS backgrounds, see [260, 343, 456] and the many excellent reviews that were written later.

Although branes have also led to new insights into 11-dimensional M-theory, the status of this theory is still rather unsettled. Therefore, this text focuses on D-branes proper as ingredients of modern string theory, the existence of which is well established, and which can be investigated in a rigorous way.

In the original formulation of D-branes by Polchinski and co-workers [119], "non-standard" boundary conditions on open strings arising from T-dualities are an essential ingredient. It is this point where the worldsheet description of string theory in terms of two-dimensional CFT comes into play. For a long time, "post-revolutionary" string literature was almost exclusively concerned with flat targets or simple variations thereof, where branes can easily be treated with methods from classical spacetime geometry. But subsequent developments revealed that naive intuitions drawn from this classical picture can be unreliable, and also that there are other kinds of branes (in particular non-BPS branes) which are not easily described in spacetime terms. Here, the worldsheet approach to D-branes, i.e. boundary CFT, proves much more effective, since it does not depend on target-space symmetries like spacetime supersymmetry. The worldsheet methods lead to a more general picture of D-branes, which appear simply as conformal boundary conditions. Geometric ideas like the interpretation of D-branes as hyperplanes in the target are no longer an essential ingredient and may sometimes in fact be misleading, even in rather simple situations.

Given the huge amount of work that has been invested into the development of the basic techniques and their applications, we have clearly been forced into many (painful) omissions. These include M-theory and the AdS/CFT correspondence, which will not be discussed, partly because, so far, worldsheet methods are only of indirect use for their study. This is not entirely true for the AdS/CFT correspondence, but the examples that can be analysed with the worldsheet techniques, such as string theory in AdS_3 without Ramond–Ramond flux, are quite exceptional. Even in this limited class of backgrounds, analytical techniques are needed to supplement the algebraic approach to be described below, because non-compactness of the curved AdS space renders spectra continuous. This provides new challenges, which, so far, have only been overcome in specific examples, model by model. In other words, in the context of the so-called non-rational CFT that describes non-compact string backgrounds, there are few model-independent constructions and results, in sharp contrast to the situation for rational CFTs that we shall treat in much detail. The interested reader can find an introduction to non-rational CFT e.g. in [409]. There is little doubt that much more technology will be developed in the future, simply because of the large number of important problems that non-rational CFT can be applied to.

There are many other applications of boundary CFT in physics and mathematics that we will only mention in passing. Among the most famous ones is the study of the Kondo effect with methods of boundary CFT that was carried out by Affleck and Ludwig [3, 4, 336]; see also [167] for related work and [458] for a discussion of the quantum wires with boundary CFT methods. Another somewhat different application concerns quantum systems with dissipation, see [91]. The connection to boundary CFT can be seen as follows: consider an open string and concentrate on one of its endpoints. Its motion will feel the presence of the rest of the string because energy can flow from the endpoint into the oscillatory modes of the string. Viewed from the endpoint, this appears as dissipation as described by the well-known Caldeira–Legget model of dissipative quantum mechanics [85]. None of these applications will be described below, but the underlying theoretical tools will be fully developed.

Let us now briefly describe the scope of this book and the organisation of its chapters, and make some suggestions on how to read them, depending on background and interests.

Chapter 1 treats free field theories on surfaces with boundaries, aiming to introduce branes without invoking abstract CFT concepts. We focus on uncompactified free bosons for the most part, but also treat fermionic theories as well as bosons compactified on a circle. To enrich the situation, non-vanishing B-fields are included from the start, which will in particular show that non-commutative geometry appears naturally in the context of boundary CFT and branes.

Chapter 2 aims at linking these field theoretic results to notions from string theory. After a lightning review of closed string theory in the first section, we turn to open strings and D-branes and try to establish a "dictionary" between worldsheet data and quantities encountered in the effective physics of strings and branes. These translation rules extend to models beyond the free field situation. We will also review some relatively technical aspects, because they are crucial in the construction of string compactifications or lead to interesting physical interpretations.

We then turn to a more general and also more abstract approach, which allows "branes" to be treated in more general situations than string theory in flat targets. To set the stage, Chapter 3 recapitulates CFT on the plane in the approach of Belavin, Polyakov and Zamolodchikov. We discuss symmetry algebras and their representations, then fields and correlation functions, including conformal Ward identities and non-linear constraints arising from sewing relations and modular invariance.

Chapter 4 contains a detailed exposition of CFT on (genus zero) surfaces with boundary, building upon the approach to CFT on the plane reviewed in the previous chapter. In the first two sections, we discuss the defining data of conformal boundary conditions and the ingredients that are new compared to CFTs on the full plane, in particular boundary fields. Section 4.3 introduces the boundary state formalism as an alternative, and computationally very efficient, description of boundary CFTs. We discuss the precise relation to the upper half-plane language used before, and also show how the examples from Chapter 1 fit into the general language. The fourth section is again devoted to non-linear constraints, some of which can be viewed as immediate generalisations of the sewing relations encountered on the plane, some of them, called Cardy constraints and exploiting modular transformations, arise naturally from the boundary state formalism. Their study provides a direct method to compute the spectrum of a boundary CFT. The collection of additional material at the end of Chapter 4 addresses crosscap states, orbifolds and a few other topics.

Chapter 5 deals with a subject that is, at least qualitatively, easier to discuss within a geometric rather than the worldsheet approach, namely D-brane moduli. We first review how to implement perturbations by arbitrary boundary fields, including relevant ones, and mention some general results like the g-theorem. We then turn to deformations by marginal boundary fields and give a criterion sufficient to show that certain operators indeed provide moduli of boundary conditions. Discussions of concrete examples and applications to string theory, including topology changes and condensation processes of brane configurations, conclude the chapter.

Chapter 6 can be seen as a "case study": we discuss the SU(2) Wess–Zumino–Witten (WZW) model and CFTs derived from it via orbifold and coset constructions, and show how to apply some of the general machinery developed in Chapters 3 to 5. In particular, we will study condensation processes of stacks of branes, and it will turn out that the necessary techniques are the same as those needed for a CFT treatment of the Kondo effect in condensed matter physics. Non-commutative worldvolumes will make another prominent appearance.

Chapter 7 is more exclusively motivated by string theory, and shows how the worldsheet formulation allows the generalisation of the notion of D-branes to string vacua formulated without recourse to a target space description, by giving a construction of BPS boundary states for Gepner models. Using the general framework of Chapter 4 and the translation rules established in Chapter 2, this task is conceptually straightforward; but the implementation of string theoretic consistency conditions (in particular the GSO projection) is somewhat technical. Gepner models provide an exact CFT description of special points in the moduli space of string compactifications whose large-volume limits correspond to Calabi–Yau manifolds in weighted projective space. We explain this relation after a review of Gepner's algebraic construction of the bulk theories, and we also discuss the geometric content of Gepner model boundary states, highlighting where the CFT results refine preconceptions based on the picture from classical geometry. The chapter concludes with an introduction to the basics of an algebraic method that allows the study of topological branes in Landau–Ginzburg models and on Calabi–Yau manifolds in a very efficient way, namely matrix factorisations.

Chapters 5 to 7 use the general formalism developed in Chapters 3 and 4, and at times language from string theory as reviewed in Chapter 2. Nevertheless, readers familiar with with boundary CFT and brane physics may have no difficulty diving directly into any of the last three chapters if they so wish.

Chapters 3 and 4 can be read as a stand-alone introduction to (boundary) CFT in the approach by BPZ and Cardy. Here and there, motivations from string theory may be felt to lurk under the surface, but they are not crucial to understanding the methods explained in these two chapters.

The opening chapters try to provide a gentle introduction to the subjects of boundary conditions and branes for readers who have some background knowledge in QFT, but little previous exposure to abstract CFT in two dimensions or to string theory.

In our research work and on the long road towards completing this book, we have greatly profited from stimulating discussions with, and valuable suggestions from, many friends and colleagues, too many to list individually. We are grateful to all of them.

London and Hamburg, November 2012

1

Free field theory with boundaries

Our aim in the first chapter is to explain the microscopic theory of strings and branes in flat backgrounds. The corresponding two-dimensional worldsheet theories can be solved with elementary methods. This will not only give us a chance to meet most of the relevant quantities from two-dimensional field theory but also to pass from worldsheet to target space concepts without much technical struggle.

Our exposition begins with an analysis of free bosonic fields on the half-plane. After a few comments on the classical action and possible boundary conditions we shall solve the model. The solution will enable us to calculate bulk operator products and correlation functions. Moreover, we shall take a first look at a few central concepts, such as gluing conditions, boundary states, bulk and boundary fields and operator products. Once free bosonic fields are dealt with, we shall perform a similar analysis for fermions. Throughout the entire chapter, some familiarity with free field theory on the complex plane will be assumed.

1.1 Free bosonic field theory

This section is devoted to the simplest two-dimensional Euclidean quantum field theory, namely a theory of free bosons. Since the model is easily solved, it allows an unobstructed approach to the main features of conformal field theory (CFT) in the presence of boundaries. The choice of boundary conditions has many interesting consequences. For example, it influences the correlation functions of bulk fields, i.e. of fields that are inserted at points in the interior of the worldsheet. Furthermore, a new set of excitations emerges that can only exist at the boundary. The precise spectrum of such boundary excitations and the associated correlations also depend on the boundary condition. In order to render our framework sufficiently rich, we shall work with $D \geq 1$ bosonic fields. Multi-component bosonic fields are also relevant for string theory where the number of bosonic fields is given by the dimension of spacetime.

1.1.1 Solution of free bosonic field theory

We want to study free bosonic field theories with a D-dimensional Euclidean target space $\mathbb{R}^D$. In order to spell out a concrete model we shall fix a constant symmetric matrix $g_{\mu\nu}$ and an antisymmetric matrix $B_{\mu\nu}$ with $\mu, \nu = 1, \ldots, D$.

These will also be referred to as *metric* and B-*field*. The D-component bosonic field $X : \Sigma \to \mathbb{R}^D$ is assumed to live on a strip $[0, \pi] \times \mathbb{R}$ or, equivalently, on the upper half-plane

$$\Sigma = \{z \in \mathbb{C} \,|\, \Im z \geq 0\} .$$

These two realisations of the worldsheet are related by the exponential map. The action of our theory is then given by the following quadratic functional,

$$S(X) = \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \; (g_{\mu\nu} + B_{\mu\nu})\, \partial X^\mu \bar{\partial} X^\nu \,, \tag{1.1}$$

where α' is a constant referred to as string tension in the string theory context. It is important to notice that for constant B the worldsheet action can be rewritten in the form

$$S(X) = \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \, g_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu + \frac{1}{\pi\alpha'} \int_\mathbb{R} dx \, B_{\mu\nu} X^\mu \partial_x X^\nu \,, \tag{1.2}$$

where the second term involving the B-field is a pure boundary term and we have used the decomposition $z = x + iy$, i.e. the coordinate x parametrises the boundary of Σ. Hence, the constant B-field does not affect the dynamics in the interior of Σ ("in the bulk"), but it provides a linear background $A_\mu(X) = B_{\mu\nu} X^\nu$ that couples only to the boundary values of the field X. If we ignore the bulk term for a moment, the boundary term alone may be interpreted as the action of a particle that moves through $\mathbb{R}^D$ in the presence of a constant magnetic field $B_{\mu\nu}$. Since the boundary coordinate x plays the role of time now, $\partial_x X^\mu$ is the velocity. If we continue to focus on the boundary, but add the bulk term back into our setup, the latter may be considered as a bath of oscillators that drags energy from the boundary. In such an interpretation, the boundary theory (1.2) becomes the starting point for the study of dissipative quantum mechanics in the presence of a magnetic background field.

The complete description of our two-dimensional field theory requires us to specify boundary conditions for X. We shall not attempt to discuss the most general such boundary conditions. With later applications in mind, we demand

$$\big(g_{\mu\nu} \partial_y X^\nu(z, \bar{z})\big)_{z=\bar{z}} = \big(i B_{\mu\nu} \partial_x X^\nu(z, \bar{z})\big)_{z=\bar{z}} \quad \text{for} \quad \mu = 1, \ldots, d \,, \tag{1.3}$$

$$\big(\partial_x X^\mu(z, \bar{z})\big)_{z=\bar{z}} = 0 \quad \text{for} \quad \mu = d+1, \ldots, D \,. \tag{1.4}$$

For $B = 0$, the first line reduces to Neumann boundary conditions for the first d components of X. A non-vanishing B-field gives rise to a deformation of Neumann boundary conditions. This has a number of interesting effects, which we shall address below. Through the second relation, we impose Dirichlet boundary conditions for the remaining $D - d$ fields, i.e. those components of X are constant along the boundary. The value of X^a at $z = \bar{z}$ shall be denoted by

$$X^a(z, \bar{z})_{z=\bar{z}} = x_0^a \quad \text{for} \quad a = d+1, \ldots, D \,.$$

The constants x_0^a define a d-dimensional hyperplane V in $\mathbb{R}^D$ through the equations $X^a = x_0^a$. From time to time we might refer to the hyperplane as a "brane". The terminology shall be explained in detail in Chapter 2.

The free bosonic field theory (1.2) with boundary conditions (1.3) and (1.4) is easy to solve. One possibility is to spell out an explicit formula for the 2-point functions from which more general correlators can then be computed via Wick's theorem. Here we shall follow an alternative operator approach.

Solution for Dirichlet boundary conditions

For the transverse directions $a = d+1, \ldots, D$, the fields X^a are constructed through the following formula for the general solution of the two-dimensional Laplace equation $\partial\bar{\partial}X^a(z,\bar{z}) = 0$ with Dirichlet boundary conditions,

$$X^a(z,\bar{z}) = x_0^a + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^a}{n} \left(z^{-n} - \bar{z}^{-n}\right) . \tag{1.5}$$

In the quantum theory, the objects α_n^a become operators obeying the relations

$$[\,\alpha_n^a, \alpha_m^b\,] = n\, g^{ab}\, \delta_{n,-m}, \ \ (\alpha_n^a)^\dagger = \alpha_{-n}^a\,. \tag{1.6}$$

The commutation relations for α_n^a ensure that the field X^a and its time derivative possess the usual canonical commutator. Reality of the bosonic field X^a is encoded in the behaviour of α_n^a under conjugation. In comparison to the solution on the full complex plane, i.e. a bulk theory with periodic boundary conditions, the construction of X^a involves a single set of modes α_n^a. While α_n^a and $\overline{\alpha}_n^a$ are independent on the complex plane, they are related by a reflection on the boundary when we pass to the half-plane.

Similar to the bulk theory, the operators α_n^a for $n \neq 0$ act as creation and annihilation operators on the Fock space $\mathcal{H}_0^a = \mathcal{V}_0$, which is generated by α_n^a with $n < 0$ from a unique ground state $|0\rangle$ subject to the conditions

$$\alpha_n^a\,|0\rangle = 0 \ \ \text{for} \ \ n > 0\,. \tag{1.7}$$

This construction of the state space $\mathcal{H}_0^a$ along with the formula (1.5) provides the complete solution for any component of the bosonic field that satisfies Dirichlet boundary conditions.

Before we turn to the boundary condition (1.3), let us briefly remark that the operators α_n^a can be obtained as Fourier modes

$$\alpha_n^a = \frac{1}{2\pi i}\int_C z^n\, J^a(z)\, dz - \frac{1}{2\pi i}\int_C \bar{z}^n\, \overline{J}^a(\bar{z})\, d\bar{z}\,. \tag{1.8}$$

Here, C is a semi-circle in Σ centred around the point $z = 0$, and $J^a, \overline{J}^a$ denote the usual chiral currents

$$J^a(z) = i\partial X^a(z,\bar{z}) = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^a\, z^{-n-1}\,,$$

$$\overline{J}^a(\bar{z}) = i\bar{\partial} X^a(z,\bar{z}) = -\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^a\, \bar{z}^{-n-1}\,.$$

Because the operators α_n^a can be constructed from the U(1) current J^a of the theory, the algebra with relations (1.6) is often referred to as U(1) current algebra. From the explicit formulas for the currents we read off that they obey

$$J^a(z) = -\overline{J}^a(\bar{z}) \tag{1.9}$$

all along the real line $z = \bar{z}$. This relation is equivalent to the Dirichlet boundary condition and it tells us that the two sets of conserved currents in our theory are identified along the real line. Hence, there is only a single set of currents living on the boundary, while there are two sets throughout the bulk of the worldsheet.

Solution for Neumann boundary conditions

Let us now repeat the above free field theory analysis for the directions subject to the boundary condition (1.3). The fields $X^i, i = 1, \ldots, d$, are once more constructed using the general solution of the wave equation

$$\begin{aligned} X^i(z,\bar{z}) = \hat{x}^i - i\sqrt{\frac{\alpha'}{2}}\,\alpha_0^i \ln z\bar{z} - i\sqrt{\frac{\alpha'}{2}}\,B^i{}_j\,\alpha_0^j \ln\frac{z}{\bar{z}} \\ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^i}{n}\left(z^{-n}+\bar{z}^{-n}\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{B^i{}_j\alpha_n^j}{n}\left(z^{-n}-\bar{z}^{-n}\right)\,, \end{aligned} \tag{1.10}$$

where summation over $j = 1, \ldots, d$ is understood. We have also raised one index of the matrix B with the help of the metric g. In passing to the quantum theory, $\hat{x}^i, \alpha_n^i$ become operators satisfying

$$[\,\alpha_n^i, \alpha_m^j\,] = n\,G^{ij}\,\delta_{n,-m}, \qquad [\,\hat{x}^i, \alpha_n^j\,] = i\sqrt{2\alpha'}\,G^{ij}\,\delta_{0,n}, \tag{1.11}$$

$$[\,\hat{x}^i, \hat{x}^j\,] = i\,\Theta^{ij}\,. \tag{1.12}$$

Furthermore, they obey the reality properties $(\hat{x}^i)^\dagger = \hat{x}^i$ and $(\alpha_n^i)^\dagger = \alpha_{-n}^i$. The commutation relations involve new structure constants G^{ij} and Θ^{ij} which are obtained from the background fields through

$$G^{ij} = \left(\frac{1}{g+B}\right)_{\mathrm{S}}^{ij}, \quad \Theta^{ij} = \left(\frac{\alpha'}{g+B}\right)_{\mathrm{A}}^{ij}. \tag{1.13}$$

Here, S or A mean that the expression in brackets gets symmetrised or antisymmetrised, respectively. By construction, G is a symmetric $d \times d$ matrix that shall serve as a new metric in many of the formulas to be derived.

Note that the matrix Θ vanishes if and only if the B-field vanishes. A non-zero Θ implies that the "centre of mass coordinates" $\hat{x}^i$ no longer commute. We shall see below that this has very interesting consequences. Readers who are unfamiliar with free field theory on the boundary are invited to set $B = 0$ upon first reading. This simplifies many of the subsequent formulas. In particular, the metric G^{ij} then agrees with the metric g^{ij} in the action. Note that the modes α_n^i