Introduction

The most basic algebraic varieties are the projective spaces, and rational varieties are their closest relatives. Rational varieties are those that are birationally equivalent to projective space. In many applications where algebraic varieties appear in mathematics, we see rational ones emerging as the most interesting examples. This happens in such diverse fields as the study of Lie groups and their representations, in the theory of Diophantine equations, and in computer-aided geometric design.

This book provides an introduction to the fascinating topic of rational, and "nearly rational," varieties. The subject has two very different aspects, and we treat them both. On the one hand, the internal geometry of rational and nearly rational varieties tends to be very rich. Their study is full of intricate constructions and surprising coincidences, many of which were thoroughly explored by the classical masters of the subject. On the other hand, to show that particular varieties are *not* rational can be a difficult problem: the classical literature is riddled with serious errors and gaps that require sophisticated general methods to repair. Indeed, only recently, with the advent of minimal model theory, have all the difficulties in classical approaches to proving nonrationality based on the study of linear systems and their singularities been ironed out.

While presenting some of the beautiful classical discoveries about the geometry of rational varieties, we pay careful attention to arithmetic issues. For example, we consider whether a variety defined over the rational numbers is rational *over* $\mathbb{Q}$, which is to say, whether there is a birational map to projective space given locally by polynomials with coefficients in $\mathbb{Q}$.

The hardest parts of the book focus on how to establish nonrationality of varieties, a difficult problem with many basic questions remaining open today. There are good general criteria, involving global differential forms, that can be used in many cases, but the situation becomes very difficult when these tests fail. For example, using simple numerical invariants called the plurigenera, it is easy to see that a smooth hypersurface in projective space whose degree exceeds its embedding dimension can not be rational. However, it is a very delicate problem to determine whether or not a lower degree hypersurface is rational.

Rationality of quadric and cubic *surfaces* was completely settled in the nineteenth century, but rationality for threefolds occupied the attention of algebraic geometers for most of the twentieth century. In the 1970s, Clemens and Griffith identified a new obstruction to rationality for a threefold inside its third topological (singular) cohomology group. This method of *intermediate Jacobians* provided the first proof that no smooth cubic threefold is rational. Because this approach fits better in a book about Hodge theory, we do not discuss it here. Instead, we prove that no smooth quartic threefold in projective four-space is rational, drawing on ideas from the minimal model program. Beyond this, very little is known: no one knows whether or not all smooth cubic fourfolds are rational, or indeed, whether there exists any nonrational smooth cubic hypersurface of any dimension greater than three.

On the other hand, in this book we do present a technique for proving nonrationality of "very general" hypersurfaces of any dimension greater than two whose degree is close to their dimension. Like other approaches to proving nonrationality, this technique uses differential forms; the novelty here is that the differential forms we use are defined on varieties of prime characteristic.

Our biggest omission is perhaps never to define precisely what we mean by a "nearly rational variety." Current research in birational algebraic geometry indicates that the most natural class of nearly rational varieties is formed by *rationally connected varieties*, introduced in Kollár *et al*. (1992). Although it is easy to state the definition, it is harder to appreciate why we claim that this is indeed the most natural class of nearly rational varieties to consider. Our aim in this book is more modest; we hope to inspire the reader to learn more about rationality questions. As a next step, we recommend the general introduction to rationally connected varieties given in Kollár (2001). Kollár (1996) contains a detailed treatment for the technically advanced.

Description of the chapters

Chapter 1 describes some basic examples of rational varieties, concentrating on quadric and cubic hypersurfaces. We give fairly complete answers for quadric hypersurfaces, but many open questions remain about cubics. We also discuss the simplest nonrationality criteria in terms of differential forms.

Cubic surfaces are examined in detail in Chapter 2. This is a classical topic that began with the works of Schläfli and Clebsch and culminated with the arithmetic studies of Segre and Manin. The main results here are about smooth cubic surfaces of Picard number one: no such cubic surface is rational. This is essentially an arithmetic result, since cubic surfaces over an algebraically closed field never have Picard number one and are always rational. On the other hand, the techniques are quite geometric, and show many of the higher dimensional methods in simpler form.

A general study of rational surfaces is given in Chapter 3. For instance, we prove Castelnuovo's criterion for rationality, giving a simple numerical characterization of rationality for smooth complex surfaces. Although this result is classical, we develop it within the modern framework of the minimal model program. This allows us to also treat surfaces that are not defined over an algebraically closed field.

In Chapter 4, we construct examples of higher dimensional smooth nonrational hypersurfaces of low degree. The constructed varieties are all Fano, which means in particular that the naive numerical invariants introduced in Chapter 1 all vanish here even though the varieties are not rational. Our proof is based on the method of reduction to prime characteristic, where we are able to exploit some of the quirks of differential forms arising from the peculiarity that the derivative of a pth power is zero in characteristic p. These positive characteristic varieties are then lifted to get examples over $\mathbb{C}$. While this method yields many examples of smooth nonrational varieties, it is not capable of producing complete families such that every smooth member is nonrational.

Chapter 5 develops the Noether–Fano method, a technique for proving nonrationality of higher dimensional varieties, analogous to the ideas presented in Chapter 2 to treat cubic surfaces. Using this approach, we produce complete families of Fano varieties in which no smooth member is rational. This example, presented in Section 5.3, is by far the simplest higher dimensional application of the Noether–Fano method. We also start the proof that no smooth quartic threefold in projective four-space is rational. This fact about quartic threefolds was first claimed by Fano (1915) although a complete proof appeared only later with the work of Iskovskih and Manin (1971).

In Chapter 6, we present more advanced machinery, namely the theory of singularities of pairs, for carrying out the general method developed in Chapter 5. Our main application is the proof of a particular numerical result which is a key ingredient in the proof that no quartic threefold is rational. These techniques also have numerous applications to diverse problems of higher dimensional geometry.

Chapter 7 contains the solutions of the exercises. The reader is strongly urged to try to work them out first instead of going to the solutions straight away.

This book began with a series of lectures by J. Kollár given at the European Mathematical Society Summer School in Algebraic Geometry in Eger, Hungary in August 1996. The notes were written up by K. E. Smith. Later new chapters were added and the old ones have been revised and reorganized. Section 4.7 (by J. Rosenberg) answers a problem raised in the original lectures.

We thank C. Araujo, A.-M. Castravet, J. Keum, S. Kovács, R. Lazarsfeld, J. Parsons, P. Vojta, and C. Werner, for many comments and corrections, N. Katz and P. Swinnerton-Dyer for some nice examples.

Prerequisites

We have devoted considerable effort to making our exposition as elementary as possible. Chapters 1 and 2 should be accessible to students who have completed a year long introductory course on classical algebraic geometry, for instance along the lines of Shafarevich (1994, vol.1). In particular, we use the language of linear systems of curves on surfaces, including their intersection theory, but we do not use cohomology.

In Chapter 3, we use basic facts about intersection theory on surfaces and cohomology for line bundles on curves and surfaces, including the Riemann–Roch theorem, Serre duality, the adjunction formula, and the Kodaira vanishing theorem. We use the most rudimentary aspects of the theory of schemes of finite type over a field in our discussion of the field of definition of a variety. Hodge theory is also mentioned in a peripheral way. Reid's lectures (1997) are an excellent and concise summary of much of the material needed in Chapter 3 and later in the book. Students familiar with Sections IV and V of Hartshorne's book (1977) should be more than adequately prepared for Chapter 3.

In Chapter 4, we work with schemes over Spec $\mathbb{Z}$ and their Kähler differentials, but we carefully explain all that is used beyond the most basic definitions. We hope that this chapter will help those familiar with classical algebraic geometry to appreciate the theory of schemes.

Chapters 5 and especially 6 are somewhat harder. We assume more sophistication in manipulating $\mathbb{Q}$-divisors, and use two major theorems that the reader is asked to accept without proof, namely the Lefschetz theorem on the Picard group of hypersurfaces and Hironaka's results on the resolution of singularities. One technically more demanding proof (of Theorem 6.32) is relegated to an Appendix. Chapter 6 may be the hardest, mainly because of the number of new concepts involved. It serves as a good introduction to some more advanced books on birational geometry or the minimal model program, for instance to Kollár and Mori (1998).

Notation and basic conventions

Let k be a field. Our main interest is in the cases $k = \mathbb{C}$ for studying geometric properties and $k = \mathbb{Q}$ for investigating the arithmetical questions. We occasionally encounter other cases too, for instance the finite fields $\mathbb{F}_q$, the real numbers $\mathbb{R}$, p-adic fields $\mathbb{Q}_p$ and algebraic number fields. The algebraic closure of k is denoted $\bar{k}$.

The notation for the ground field is suppressed when the field is clear from the context or irrelevant, but occasionally we write X_k to emphasize that the variety X is defined over the ground field k. If $L \supset k$ is a field extension then X_L denotes the variety X_k viewed as being over L. Technically speaking, $X_L = X_k \times_{\operatorname{Spec} k} \operatorname{Spec} L$.

If L is any field containing k, then an *L-point*, or an *L-rational point*, is one having all of its coordinates defined over L. That is, thinking of a variety over k as locally a subvariety of $\mathbb{A}^n$ given by the vanishing of polynomials with coefficients in k, then an L-rational point is given by an n-tuple of elements of L satisfying the defining polynomials. Thinking more scheme-theoretically, an L-point on a scheme X can be defined as a morphism $\operatorname{Spec} L \to X$. In particular, a k-rational point on a k-scheme corresponds to a maximal ideal whose residue field is k. The symbol $X(L)$ denotes the set of L-points of X.

Morphisms and *rational maps* between varieties are always assumed to be defined over the ground field, except where explicitly stated otherwise. Likewise, linear systems on a variety X_k are assumed defined over k.

Morphisms are denoted by solid arrows $\to$ and rational maps by dashed arrows $\dashrightarrow$. The "image" of a rational map is the closure of the image of the morphism obtained by restricting the rational map to some nonempty open set where it is defined; in the same way, we define the image of a subvariety under a rational map, provided that the map is defined at its generic point. In particular, let $f : Y \dashrightarrow X$ be a rational map and suppose that f is defined at the generic point of some subvariety Z of Y. Then the image of Z on X, denoted $f_*(Z)$, is the closure in X of the set $f|_{X_0}(Z \cap X_0)$, where X_0 is some open set meeting Z on which f is a well-defined morphism. In the case of birational maps, the image is also called the *birational transform*, especially in the case where this image has the same dimension.

Let X be a normal variety. An irreducible and reduced subscheme of codimension one is called a *prime divisor*. A *divisor* on X is a formal linear combination $D = \sum d_i D_i$ of prime divisors where $d_i \in \mathbb{Z}$. In using this notation we assume that the D_i are distinct. A *$\mathbb{Q}$-divisor* is a formal linear combination $D = \sum d_i D_i$ of prime divisors where $d_i \in \mathbb{Q}$. The divisor D is called *effective* if $d_i \geq 0$ for every i. A divisor (or $\mathbb{Q}$-divisor) D is called *$\mathbb{Q}$-Cartier* if mD is

Cartier for some nonzero integer m, where by Cartier we mean that it is locally defined by a single equation. On a smooth variety every divisor is Cartier. The *support* of $D = \sum d_i D_i$, denoted by Supp D is the subscheme $\cup_{d_i \neq 0} D_i$. *Linear equivalence* of two divisors is denoted by $D_1 \sim D_2$.

A property of a variety X_k refers to the variety considered over k. We add the adjective *geometrically* when talking about a property of $X_{\bar{k}}$. For example, the affine plane curve defined by the equation $x^2 + y^2 = 0$ is irreducible as a $\mathbb{Q}$-variety but it is geometrically reducible. For many properties (including smoothness or projectivity), the distinction does not matter.

Varieties are assumed reduced and irreducible, except where explicitly stated otherwise. In particular, the terms "smooth curve" and "smooth surface" always refer to *connected* smooth surfaces and curves. The one exception is that we use the term "curve on a surface" to mean any effective divisor, which may or may not be reduced and irreducible. Because we are concerned with birational properties, there is no loss of generality in assuming all varieties to be quasi-projective. In any case, our main interest is in smooth projective varieties.

In writing these notes, our policy was not to be sidetracked by anomalies in positive characteristic. These usually appear when the base field is not perfect, that is, when it has algebraic extensions obtained by taking pth roots (in characteristic p). Technical problems related to such issues are relegated to exercises and they can be safely ignored for most of the book.

The exception is Chapter 4 where the unusual properties of such field extensions are exploited to prove several results about varieties over $\mathbb{C}$ or $\mathbb{Q}$.

1

Examples of rational varieties

In this chapter, we introduce rational varieties through examples. After giving the fundamental definitions in the first section and settling the rationality question for curves in Section 2, we continue with the rich theory of quadric hypersurfaces in Section 3. This is essentially a special case of the theory of quadratic forms, though the questions tend to be strikingly different.

Quadrics over finite fields are discussed in Section 4. Several far-reaching methods of algebraic geometry appear here in their simplest form.

Cubic hypersurfaces are much more subtle. In Section 5, we discuss only the most basic rationality and unirationality facts for cubics. A further smattering of rational varieties is presented in Section 6, together with a more detailed look at determinantal representations for cubic surfaces.

A very general and useful nonrationality criterion, using differential forms, is discussed in Section 7.

1.1 Rational and unirational varieties

Roughly speaking, a variety is *unirational* if a dense open subset is parametrized by projective space, and *rational* if such a parametrization is one-to-one.

To be precise, fix a ground field k, and let X be a variety defined over k. It is important to bear in mind that k need not be algebraically closed and that all constructions involving the variety X are carried out over the ground field k.

Definition 1.1. A variety is *rational* if it is birationally equivalent to projective space. Explicitly, the variety X is rational if there exists a birational map $\mathbb{P}^n \dashrightarrow X$.

Definition 1.2. The variety X is *unirational* if there exists a generically finite dominant map $\mathbb{P}^n \dashrightarrow X$.

Rational varieties were once called "birational," in reference to the rational maps between them and projective space in each direction. "Unirationality" thus refers to the map from $\mathbb{P}^n$ to the variety, defined in one direction only. This explains the odd use of the prefix "uni" in referring to a map which is finite-to-one.

We emphasize that in both definitions above, the varieties and the maps are defined over the fixed ground field k. This means that the variety X is defined locally by polynomials with coefficients in k, and also that the map can be described by polynomials with coefficients in k.

Our guiding question throughout this book is the following: *Which varieties are rational or unirational*?

The rationality or unirationality of a variety may depend subtly on the field of definition. For example, a variety defined over $\mathbb{Q}$ may be considered as a variety defined over $\mathbb{R}$. It is possible that there is a birational map given by polynomials with *real* coefficients from projective space to the variety, but there is no such birational map given by polynomials with *rational* coefficients. Our first example nicely illustrates this point.

EXAMPLE 1.3. Consider the plane conic C defined by the homogeneous equation $x^2 + y^2 = pz^2$, where p is a prime number congruent to -1 modulo 4. We claim that

1. the Diophantine equation $x^2 + y^2 = pz^2$ has no rational solutions (aside from the trivial solution $x = y = z = 0$),
2. the curve C is not rational over $\mathbb{Q}$, and
3. the curve C is rational over $\mathbb{Q}(\sqrt{p})$.

Indeed, assume that $x^2 + y^2 = pz^2$ has a rational solution. By clearing denominators, we may assume that x, y, and z are integers, not all divisible by p. If neither x nor y is divisible by p, then the congruence $x^2 \equiv -y^2 \mod p$ leads to a solution of $u^2 \equiv -1 \mod p$. But this is impossible since $p \equiv -1 \mod 4$. (This easy fact is sometimes called Euler's criterion for quadratic congruences; if you have not seen it before, check by hand the examples $p = 3, 7, 11$ before looking it up in any elementary number theory book.) This contradiction forces p to divide both x and y. But then p^2 divides pz^2, so that p divides z as well, a contradiction. This establishes (1).

Now, if C is rational (or even unirational) over $\mathbb{Q}$, then images of the rational points under the map $\mathbb{P}^1 \dashrightarrow C$ give plenty of rational points on C, contradicting (1).

Finally, (3) can be seen from the explicit parametrization

$$\mathbb{P}^1 \dashrightarrow C \subset \mathbb{P}^2$$

given by

$$(t:1) \mapsto (t^2-1:2t:\frac{1}{\sqrt{p}}(t^2+1)).$$

This is a special case of the parametrization given later in the proof of Theorem 1.11 for a general quadric hypersurface.

We say that X is *geometrically rational* if X is rational over $\bar{k}$. The reader is cautioned however, that the literature is inconsistent: some authors use the term "rational" to mean "geometrically rational."

One must be careful about trusting intuition based on extensive study of algebraic varieties over an algebraically closed field. For example, even when a map $\mathbb{P}^n \dashrightarrow X$ as in Definition 1.2 is dominant, the induced map on the set of k-points $\mathbb{P}^n(k) \dashrightarrow X(k)$ can be very far from surjective. For instance, with the ground field fixed to be $\mathbb{Q}$, consider the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $(s:t) \mapsto (s^2:t^2)$. The image of the rational points is a very sparse subset of the set of all rational points of the target variety. This is typical for maps defined over algebraically non-closed fields.

1.2 Rational curves

Over $\mathbb{C}$, and more generally, over any algebraically closed field, the only smooth projective curve remotely resembling the projective line is $\mathbb{P}^1$ itself. Indeed, as is frequently covered in a first course in algebraic geometry, the following are equivalent for a smooth projective curve over an algebraically closed field:

1. the curve is isomorphic to the projective line;
2. the curve is birationally equivalent to the projective line;
3. there is a nonconstant map from the projective line to the curve;
4. the curve has no nonzero global holomorphic (that is, Kähler) one-forms: in other words, the canonical linear system is empty.

But what about curves over algebraically non-closed fields? It is still the case that every rational map from a curve is, in fact, an everywhere-defined morphism; the usual proof of this fact does not require an algebraically closed ground field. So over any ground field, a birational map from a curve is an isomorphism, and (1) and (2) are equivalent. In this section, we see also that (3) is equivalent to (1) and to (2) over an arbitrary ground field, but that (4) is not. In fact, we see that rationality questions for curves come down to the case of plane conics, where the answers depend on the ground field.

Given a smooth projective curve, how can we tell if it is rational? Of course, if a curve is rational over k, it is certainly rational over its algebraic closure $\bar{k}$, so we might as well restrict our attention to geometrically rational curves. Among all the different representations of a smooth geometrically rational curve (for instance, as a projective line, a plane conic, a twisted cubic, and so on), the following proposition shows that the plane conics account for all possible birational models of the projective line over any field.

PROPOSITION 1.4. *A smooth projective geometrically rational curve is isomorphic to a smooth plane conic.*

Again we emphasize (we will soon stop!) that the interesting part of this statement is that all this is going on over some fixed ground field k, which need not be algebraically closed. So any smooth curve over k that is rational when considered as a variety over $\bar{k}$ must be isomorphic (over k) to a curve in $\mathbb{P}^2$ defined by a quadratic polynomial with coefficients in the ground field k. This would be obvious if k were algebraically closed.

The proof uses two basic results of algebraic geometry over algebraically nonclosed fields. Both are quite elementary but they do not always receive the emphasis that they deserve in introductory texts.

PROPOSITION 1.5. *Let X be a smooth quasi-projective variety defined over a field k. Then it has a canonical divisor defined over k. Thus we can speak of the canonical divisor class K_X as a linear equivalence class defined over k.*

PROOF. Let us start with the most classical case when k has characteristic zero and X is a curve.

If g is any function on X, the divisor of dg is a canonical divisor. If g is in $k(X)$ then the corresponding divisor (dg) is defined over k.

We do something similar in higher dimensions. Choose $g_1, \ldots, g_n$ algebraically independent functions of $k(X)$. Then the divisor of $dg_1 \wedge \cdots \wedge dg_n$ is a canonical divisor defined over k.

We have to be a little more careful in positive characteristic. The problem is that if g is a pth power then $dg = 0$ and its divisor (dg) is not defined. It is not hard to show that this problem can be avoided by a careful choice of the functions g_i. See, for instance, van der Waerden (1991, 19.7).

Another possibility, more in keeping with modern techniques, is to construct the sheaf of differential forms (i.e. the sheaf of Kähler differential one-forms) as in Shafarevich (1994, III.5) and define the canonical class as the divisor class corresponding to its determinant bundle. □

PROPOSITION 1.6. *Let D be a divisor on a smooth projective variety X defined over a field k. Then the dimension of the complete linear system defined*