Cambridge University Press & Assessment
978-0-521-83197-0 — Introduction to Foliations and Lie Groupoids
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Prerequi

In this book we presuppose some familiarity with the basic notions of differential topology and geometry. Good references are e.g. Guillemin–Pollack (1974) and Bott–Tu (1982). We shall list some of these notions, partly to fix the notations.

Recall that a *smooth manifold* (or a C^∞-manifold) of dimension n (where $n = 0, 1, \ldots$) is a second-countable Hausdorff space M, together with a maximal *atlas* of open embeddings (*charts*)

$$(\varphi_i \colon U_i \longrightarrow \mathbb{R}^n)_{i \in I}$$

of open subsets $U_i \subset M$ into $\mathbb{R}^n$, such that $M = \bigcup_{i \in I} U_i$ and the *change-of-charts homeomorphisms*

$$\varphi_{ij} = \varphi_i \circ (\varphi_j|_{U_i \cap U_j})^{-1} \colon \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$

are smooth, for any $i \quad \in I$. Note that these satisfy the cocycle condition $\varphi_{ij}(\varphi_{jk}(x)) = \varphi_{ik}(x)$, $x \in \varphi_k(U_i \cap U_j \cap U_k)$. There is an associated notion of a *smooth* map between smooth manifolds. Any smooth manifold (Hausdorff and second-countable) is paracompact, which is sufficient for the existence of partitions of unity.

The notions of (maximal) atlas and of smooth map also make sense if M is any topological space, not necessarily second countable or Hausdorff. We refer to such a space with a maximal atlas as a *non-Hausdorff manifold* or a *non-second-countable manifold*. There are many more non-Hausdorff manifolds than the usual Hausdorff ones, even in dimension 1 (see Haefliger–Reeb (1957)). We shall have occasion to consider such non-Hausdorff manifolds later in this book.

The reader should be familiar with the notion of the *tangent bundle* $T(M)$ of M, which is a vector bundle over M of rank n, where n is the dimension of the manifold M. The *tangent space* $T_x(M)$ of M at

$x \in M$ is the fibre of $T(M)$ over x. The (smooth) sections of the tangent bundle $T(M)$ are the *vector fields* on M. The $C^\infty(M)$-module $\mathfrak{X}(M)$ of all vector fields on M is a Lie algebra, and the Lie bracket on $\mathfrak{X}(M)$ satisfies the Leibniz identity

$$[X, f\quad] = f[X, Y] + X(f)Y$$

for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

Also, we have the space $\Omega^k(M)$ of *differential* k*-forms* on M, for any $k = 0, 1, \ldots, n$, with *exterior differentiation* $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ and *exterior product* $\wedge\colon \Omega^k(M) \otimes \Omega^l(M) \to \Omega^{k+l}(M)$. With this, $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$ becomes a differential graded algebra, which is commutative (in the graded sense). The cohomology of $(\Omega(M), d)$ is called the *de Rham cohomology* of M, and denoted by

$$H_{\mathrm{dR}}(M) = \bigoplus_{k=0}^{n} H^k_{\mathrm{dR}}(M) \; .$$

A smooth map $f\colon M \to N$ between smooth manifolds has a *derivative* $d\quad\colon T(M) \to T(N)$, which is a bundle map over f. The derivative of f at $x \in M$ is the restriction of $d\quad$ to the corresponding tangent spaces over x and $f(x)$, and denoted by $(d\quad)_x\colon T_x(M) \to T_{f(x)}(N)$. The map f is an *immersion* if each $(d\quad)_x$ is injective, and a *submersion* if each $(d\quad)_x$ is surjective. These have canonical local forms on a small neighbourhood of $x \in M$:

(i) If f is an immersion, there exist open neighbourhoods $U \subset M$ of x and $V \subset N$ of $f(x)$ with $f(U) \subset V$ and diffeomorphisms $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^p$ such that

$$(\psi \circ f \circ \varphi^{-1})(y) = (y\ \ 0)$$

with respect to the decomposition $\mathbb{R}^p = \mathbb{R}^n \times \mathbb{R}^{p-n}$.

(ii) If f is a submersion, there exist open neighbourhoods $U \subset M$ of x and $V \subset N$ of $f(x)$ with $f(U) = V$ and diffeomorphisms $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^p$ such that

$$(\psi \circ f \circ \varphi^{-1})(y\quad) = y$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$.

A smooth map $f\colon M \to N$ is a *diffeomorphism* if it is a bijection and has a smooth inverse. The map f is a *local diffeomorphism* (or *étale map*) if $(d\quad)_x$ is an isomorphism for any $x \in M$. Any bijective local diffeomorphism is a diffeomorphism.

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A smooth map $g\colon K \to N$ is said to be an *embedding* if it is an immersion and a topological embedding. This makes K a *submanifold* of N, and $T(K)$ a subbundle of $T(N)$.

If K is a submanifold of N and $f\colon M \to N$ a smooth map, one says that f is *transversal* to K if $(d\ )_x(T_x(M)) + T_{f(x)}(K) = T_{f(x)}(N)$ for every $x \in f^{-1}(K)$.

For every submanifold K of N there exists an open neighbourhood $U \subset N$ of K which has the structure of a vector bundle over K, with the inclusion $K \hookrightarrow U$ corresponding to the zero section. In particular, the projection $U \to K$ of this bundle is a retraction. Such a U is called a *tubular neighbourhood* of K.

Recall that, on a vector bundle E of rank n over a manifold M, one can always choose a *Riemannian structure* (by using partitions of unity). A *Riemannian metric* on M is a Riemannian structure on $T(M)$. The structure group of E can be reduced to $O(n)$. The bundle E is called *orientable* if its structure group can be reduced to $SO(n)$. An *orientation* of an orientable vector bundle E is an equivalence class of oriented trivializations of E.

1

Fol

Intuitively speaking, a foliation of a manifold M is a decomposition of M into immersed submanifolds, the leaves of the foliation. These leaves are required to be of the same dimension, and to fit together nicely.

Such foliations of manifolds occur naturally in various geometric contexts, for example as solutions of differential equations and integrable systems, and in symplectic geometry. In fact, the concept of a foliation first appeared explicitly in the work of Ehresmann and Reeb, motivated by the question of existence of completely integrable vector fields on three-dimensional manifolds. The theory of foliations has now become a rich and exciting geometric subject by itself, as illustrated be the famous results of Reeb (1952), Haefliger (1956), Novikov (1964), Thurston (1974), Molino (1988), Connes (1994) and many others.

We start this book by describing various equivalent ways of defining foliations. A foliation on a manifold M can be given by a suitable foliation atlas on M, by an integrable subbundle of the tangent bundle of M, or by a locally trivial differential ideal. The equivalence of all these descriptions is a consequence of the Frobenius integrability theorem. We will give several elementary examples of foliations. The simplest example of a foliation on a manifold M is probably the one given by the level sets of a submersion $M \to N$. In general, a foliation on M is a decomposition of M into leaves which is *locally* given by the fibres of a submersion.

In this chapter we also discuss some first properties of foliations, for instance the property of being orientable or transversely orientable. We show that a transversely orientable foliation of codimension 1 on a manifold M is given by the kernel of a differential 1-form on M, and that this form gives rise to the so-called Godbillon–Vey class. This is a class of degree 3 in the de Rham cohomology of M, which depends only on the foliation and not on the choice of the specific 1-form. Furthermore, we discuss here several basic methods for constructing foliations. These include the product and pull-back of foliations, the formation of foliations on quotient manifolds, the construction of foliations by 'suspending' a diffeomorphism or a group of diffeomorphisms, and foliations associated to actions of Lie groups.

1.1 Definition and first examples

Let M be a smooth manifold of dimension n. A *foliation atlas* of codimension q of M (where $0 \leq q \leq n$) is an atlas

$$(\varphi_i \colon U_i \longrightarrow \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q)_{i \in I}$$

of M for which the change-of-charts diffeomorphisms φ_{ij} are locally of the form

$$\varphi_{ij}(x,y) = (g_{ij}(x,y), h_{ij}(y))$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$. The charts of a foliation atlas are called the *foliation charts*. Thus each U_i is divided into *plaques*, which are the connected components of the submanifolds $\varphi_i^{-1}(\mathbb{R}^{n-q} \times \{y\})$, $y \in \mathbb{R}^q$, and the change-of-charts diffeomorphisms preserve this division (Figure 1.1). The plaques globally amalgamate

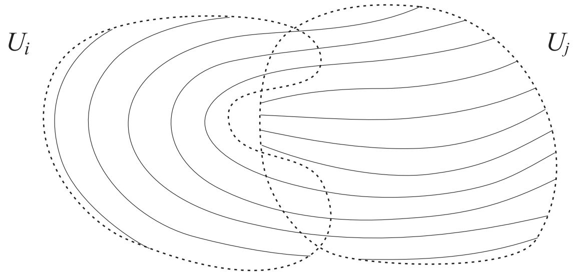

Fig. 1.1. Two foliation charts

into *leaves*, which are smooth manifolds of dimension $n - q$ injectively immersed into M. In other words, two points $x, y \in M$ lie on the same leaf if there exist a sequence of foliation charts $U_1, \ldots, U_k$ and a sequence of points $x = p_0, p_1, \ldots, p_k = y$ such that p_{j-1} and p_j lie on the same plaque in U_j, for any $1 \leq j \leq k$.

A *foliation* of *codimension* q of M is a maximal foliation atlas of M of codimension q. Each foliation atlas determines a foliation, since it is included in a unique maximal foliation atlas. Two foliation atlases define the same foliation of M precisely if they induce the same partition of M into leaves. A (smooth) *foliated manifold* is a pair $(M, \mathcal{F})$, where M is a smooth manifold and $\mathcal{F}$ a foliation of M. The *space of leaves* $M/\mathcal{F}$ of a foliated manifold $(M, \mathcal{F})$ is the quotient space of M, obtained by identifying two points of M if they lie on the same leaf of $\mathcal{F}$. The dimension of $\mathcal{F}$ is $n - q$. A (smooth) map between foliated manifolds $f\colon (M, \mathcal{F}) \to (M', \mathcal{F}')$ is a (smooth) map $f\colon M \to N$ which preserves the foliation structure, i.e. which maps leaves of $\mathcal{F}$ into the leaves of $\mathcal{F}'$.

This is the first definition of a foliation. Instead of smooth foliations one can of course consider C^r-foliations, for any $r \in \{0, 1, \ldots, \infty\}$, or (real) analytic foliations. Standard references are Bott (1972), Hector–Hirsch (1981, 1983), Camacho–Neto (1985), Molino (1988) and Tondeur (1988). In the next section we will give several equivalent definitions: in terms of a Haefliger cocycle, in terms of an integrable subbundle of $T(M)$, and in terms of a differential ideal in $\Omega(M)$. But first we give some examples.

Examples 1.1 (1) The space $\mathbb{R}^n$ admits the *trivial* foliation of codimension q, for which the atlas consists of only one chart id: $\mathbb{R}^n \to \mathbb{R}^{n-q} \times \mathbb{R}^q$. Of course, any linear bijection $A\colon \mathbb{R}^n \to \mathbb{R}^{n-q} \times \mathbb{R}^q$ determines another one whose leaves are the affine subspaces $A^{-1}(\mathbb{R}^{n-q} \times \{y\})$.

(2) Any submersion $f\colon M \to N$ defines a foliation $\mathcal{F}(f)$ of M whose leaves are the connected components of the fibres of f. The codimension of $\mathcal{F}(f)$ is equal to the dimension of N. An atlas representing $\mathcal{F}(f)$ is derived from the canonical local form for the submersion f. Foliations associated to the submersions are also called *simple* foliations. The foliations associated to submersions with connected fibres are called *strictly simple*. A simple foliation is strictly simple precisely when its space of leaves is Hausdorff.

(3) (Kronecker foliation of the torus) Let a be an irrational real number, and consider the submersion $s\colon \mathbb{R}^2 \to \mathbb{R}$ given by $s(x, y) = x - a$. By (2) we have the foliation $\mathcal{F}(s)$ of $\mathbb{R}^n$. Let $f\colon \mathbb{R}^2 \to T^2 = S^1 \times S^1$ be the standard covering projection of the torus, i.e. $f(x, y) = (\mathrm{e}^{2\pi \mathrm{i} x}, \mathrm{e}^{2\pi \mathrm{i} y})$. The foliation $\mathcal{F}(s)$ induces a foliation $\mathcal{F}$ of T^2: if φ is a foliation chart for $\mathcal{F}(s)$ such that $f|_{\mathrm{dom}\varphi}$ is injective, then $\varphi \circ (f|_{\mathrm{dom}\varphi})^{-1}$ is a foliation chart for $\mathcal{F}$. Any leaf of $\mathcal{F}$ is diffeomorphic to $\mathbb{R}$, and is dense in T^2 (Figure 1.2).

(4) (Foliation of the Möbius band) Let $f\colon \mathbb{R}^2 \to M$ be the standard covering projection of the (open) Möbius band: $f(x, y) = f(x', y')$

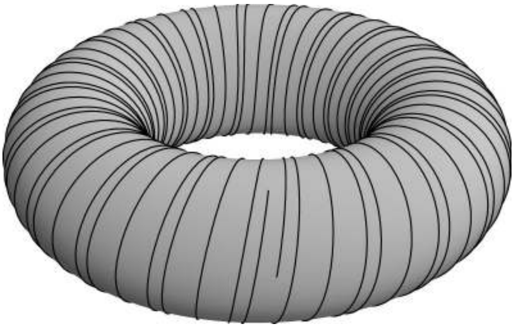

Fig. 1.2. Kronecker foliation of the torus

precisely if $x' - x \in \mathbb{Z}$ and $y' = (-1)^{x'-x} y$. The trivial foliation of codimension 1 of $\mathbb{R}^2$ induces a foliation $\mathcal{F}$ of M, in the same way as in (3). All the leaves of $\mathcal{F}$ are diffeomorphic to S^1, and they are wrapping around M twice, except for the 'middle' one: this one goes around only once (Figure 1.3).

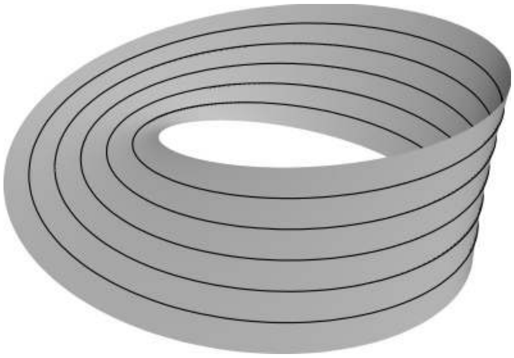

Fig. 1.3. Foliation of the Möbius band

(5) (The Reeb foliation of the solid torus and of S^3) One can also define the notion of a foliation of a manifold with boundary in the obvious way; however, one usually assumes that the leaves of such a foliation behave well near the boundary, by requiring either that they are transversal to the boundary, or that the connected components of the boundary are leaves. An example of the last sort is the Reeb foliation of the solid torus, which is given as follows.

Consider the unit disk $D = \{ z \mid z \in \mathbb{C}, |z| \leq 1 \}$, and define a submer-

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sion $f\colon \mathrm{Int}(D) \times \mathbb{R} \to \mathbb{R}$ by

$$f(z\quad) = \mathrm{e}^{\frac{1}{1-|z|^2}} - x \,.$$

So we have the foliation $\mathcal{F}(f)$ of $\mathrm{Int}(D) \times \mathbb{R}$, which can be extended to a foliation of the cylinder $D \times \mathbb{R}$ by adding one new leaf: the boundary $S^1 \times \mathbb{R}$. Now $D \times \mathbb{R}$ is a covering space of the solid torus $X = D \times S^1$ in the canonical way, and the foliation of $D \times \mathbb{R}$ induces a foliation of the solid torus. We will denote this foliation by $\mathcal{R}$. The boundary torus of this solid torus is a leaf of $\mathcal{R}$. Any other leaf of $\mathcal{R}$ is diffeomorphic to $\mathbb{R}^2$, and has the boundary leaf as its set of adherence points in X. The *Reeb foliation* of X is any foliation $\mathcal{F}$ of X of codimension 1 for which there exists a homeomorphism of X which maps the leaves of $\mathcal{F}$ onto the leaves of $\mathcal{R}$ (Figure 1.4).

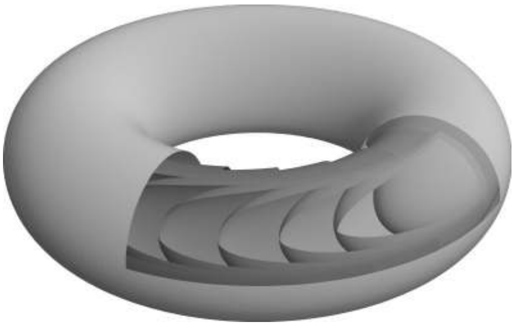

Fig. 1.4. The Reeb foliation of the solid torus

The three-dimensional sphere S^3 can be decomposed into two solid tori glued together along their boundaries, i.e.

$$S^3 \cong X \cup_{\partial X} X \,.$$

Since $\partial\quad$ is a leaf of the Reeb foliation of X, we can glue the Reeb foliations of both copies of X along $\partial\quad$ as well. This can be done so that the obtained foliation of S^3 is smooth. This foliation has a unique compact leaf and is called the *Reeb foliation* of S^3.

Exercise 1.2 Describe in each of these examples explicitly the space of leaves of the foliation. (You will see that this space often has a very poor structure. Much of foliation theory is concerned with the study of 'better models' for the leaf space.)

1.2 Alternative definitions of foliations

A foliation $\mathcal{F}$ of a manifold M can be equivalently described in the following ways (here n is the dimension of M and q the codimension of $\mathcal{F}$).

(i) By a foliation atlas $(\varphi_i : U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ of M for which the change-of-charts diffeomorphisms φ_{ij} are *globally* of the form $\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y))$ with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$.

(ii) By an open cover (U_i) of M with submersions $s_i \colon U_i \to \mathbb{R}^q$ such that there are diffeomorphisms (necessarily unique)

$$\gamma_{ij} \colon s_j(U_i \cap U_j) \longrightarrow s_i(U_i \cap U_j)$$

with $\gamma_{ij} \circ s_j|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}$. (The diffeomorphisms γ_{ij} satisfy the cocycle condition $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$. This cocycle is called the *Haefliger cocycle* representing $\mathcal{F}$.)

(iii) By an *integrable subbundle* E of $T(M)$ of rank $n - q$. (Here integrable (or involutive) means that E is closed under the Lie bracket, i.e. if $X, Y \in \mathfrak{X}(M)$ are sections of E, then the vector field $[X, Y]$ is also a section of E.)

(iv) By a locally trivial *differential* (graded) *ideal* $\mathcal{J} = \bigoplus_{k=1}^n \mathcal{J}^k$ of rank q in the differential graded algebra $\Omega(M)$. (An ideal $\mathcal{J}$ is locally trivial of rank q if any point of M has an open neighbourhood U such that $\mathcal{J}|_U$ is the ideal in $\Omega(M)|_U$ generated by q linearly independent 1-forms. An ideal $\mathcal{J}$ is differential if $d\mathcal{J} \subset \mathcal{J}$.)

Before we go into details of why these descriptions of the concept of foliation are equivalent, we should point out that the bundle E of (iii) consists of tangent vectors to M which are tangent to the leaves, while a differential k-form is in the ideal $\mathcal{J}$ of (iv) if it vanishes on any k-tuple of vectors which are all tangent to the leaves.

Ad (i): Any foliation atlas $(\varphi_i \colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ of $\mathcal{F}$ has a refinement which satisfies the condition in (i). To see this, we may first assume that (U_i) is a locally finite cover of M. Next, we may find a locally finite refinement (V_k) of (U_i) such that $V_k \cup V_l$ is contained in some U_i for any non-disjoint V_k and V_l. As any V_k is contained in a U_{i_k}, we may take $\psi_k = \varphi_{i_k}|_{V_k}$. Further we may choose each V_k so small that for any $U_j \supset V_k$, the change-of-charts diffeomorphism $\varphi_j \circ \psi_k^{-1}$ is globally of the form $(g_{jk}(x, y), h_{jk}(y))$, and that h_{jk} is an embedding. This refinement (ψ_k) of (φ_i) is a foliation atlas of M which satisfies the condition in (i).

Ad (ii): If (U_i, s_i, γ_{ij}) is a Haefliger cocycle on M, choose an atlas $(\varphi_k\colon V_k \to \mathbb{R}^n)$ so that each V_k is a subset of an U_{i_k} and φ_k renders s_{i_k} in the normal form for a submersion: it is surjective, and there exists a diffeomorphism $\psi_k\colon s_{i_k}(V_k) \to \mathbb{R}^q$ such that $\psi_k \circ s_{i_k} = \mathrm{pr}_2 \circ \varphi_k$. This is a foliation atlas of the form in (i): if $(x, y) \in \varphi_k(V_k \cap V_l) \subset \mathbb{R}^{n-q} \times \mathbb{R}^q$, we have

$$\begin{aligned}(\mathrm{pr}_2 \circ \varphi_l \circ \varphi_k^{-1})(x, y) &= (\psi_l \circ s_{i_l} \circ \varphi_k^{-1})(x, y)\\ &= (\psi_l \circ \gamma_{i_l i_k} \circ s_{i_k} \circ \varphi_k^{-1})(x, y)\\ &= (\psi_l \circ \gamma_{i_l i_k} \circ \psi_k)(y)\ .\end{aligned}$$

Conversely, if $(\varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$ is a foliation atlas of the form in (i), take $s_i = \mathrm{pr}_2 \circ \varphi_i$ and $\gamma_{ij} = h_{ij}$. This gives a Haefliger cocycle on M which represents the same foliation.

Ad (iii): Let us assume that the foliation is given by a foliation atlas $(\varphi_i\colon U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)$. Define a subbundle E of $T(M)$ locally over U_i by

$$E|_{U_i} = \mathrm{Ker}(d(\mathrm{pr}_2 \circ \varphi_i))\ ,$$

i.e. by the kernel of the $\mathbb{R}^q$-valued 1-form $\alpha = d(\mathrm{pr}_2 \circ \varphi_i)$. For any such a 1-form and any vector fields X, Y on U_i we have $2d\ \ (X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$. Since our α is closed, it follows that

$$\alpha([X, Y]) = X(\alpha(Y)) - Y(\alpha(X))\ .$$

Using this it is clear that E is an integrable subbundle of $T(M)$ of codimension q.

The bundle E is uniquely determined by the foliation $\mathcal{F}$: a tangent vector $\xi \in T_x(M)$ is in E precisely if ξ is tangent to the leaf of L through x. The bundle E is called the *tangent bundle* of $\mathcal{F}$, and is often denoted by $T(\mathcal{F})$. A section of $T(\mathcal{F})$ is called a vector field *tangent* to $\mathcal{F}$. The Lie algebra $\Gamma(T(\mathcal{F}))$ of sections of $T(\mathcal{F})$ will also be denoted by $\mathfrak{X}(\mathcal{F})$.

Conversely, an integrable subbundle E of codimension q of $T(M)$ can be locally integrated (Frobenius theorem, see Appendix of Camacho–Neto (1985)): for any point $x \in M$ there exist an open neighbourhood $U \subset M$ and a diffeomorphism $\varphi\colon U \to \mathbb{R}^{n-q} \times \mathbb{R}^q$ such that $E|_U = \mathrm{Ker}(d(\mathrm{pr}_2 \circ \varphi))$. By using these kinds of diffeomorphisms as foliation charts, one obtains a foliation atlas of the foliation.

Ad (iv): For any subbundle E of $T(M)$, define the (graded) ideal $\mathcal{J} = \bigoplus_{k=1}^n \mathcal{J}^k$ in $\Omega(M)$ as follows: for $\omega \in \Omega^k(M)$,

$$\omega \in \mathcal{J}^k \text{ if and only if}$$