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Introduction

Many problems of practical interest involve non-linear behavior of solids and structures. In the present context a solid means a body with a firm shape, as opposed to a fluid, while a structure refers to a solid composed of slender elements such as beams, plates and shells. Typical problems are the motion of robots, collapse scenarios of structures, metal forming processes in industrial production, and material deformation and failure in geotechnical engineering. These problems typically involve a considerable change of shape, often accompanied by non-linear material behavior.

The finite element method is an important tool for the analysis of non-linear problems, such as geometrical and material non-linear behavior of solids and structures. The solution of non-linear problems by the finite element method involves modeling, leading to the formulation of an appropriate set of non-linear equations describing the problem, followed by an appropriate strategy for the numerical solution of these equations. In contrast to linear problems, where the solution strategy reduces to the solution of a system of linear equations, the solution phase in a non-linear problem typically involves an iterative procedure.

Non-linear modeling and analysis is a very active research area with many engineering applications. The many different aspects involved are not covered in any single text. However, some central references to general texts should be given here. A brief introduction to some of the basic problems of non-linear finite element analysis of solids and structures is included in the book by Cook *et al.* (1989). A general state-of-the-art presentation of the finite element method, including the non-linear aspects of solids, structures and fluids, has been given in Zienkiewicz and Taylor (2000). A presentation with main emphasis on incremental formulation of geometrically non-linear problems, including details of implementation, has been given by Bathe (1996). The books by Crisfield (1991, 1997) and Belytschko *et al.*

(2000) are entirely devoted to non-linear analysis of solids and structures, combining illustrative examples with specific finite element procedures.

The present text is an introduction to some of the central ideas of non-linear modeling and finite element analysis. It covers theoretical aspects of geometric and material non-linearity and associated numerical techniques. The text proceeds from the elementary level to a fairly rigorous presentation of ideas used in current research. Only the main ideas can be covered, and the references should be consulted according to need. This first chapter gives an illustration of geometric non-linear behavior with reference to a simple two-element truss model. The example serves as a vehicle for an informal introduction to a non-linear load–displacement relation, the tangent stiffness, and the relation between the equilibrium and the virtual work approach to the problem. The example also provides a simple realistic non-linear equation on which to try different variants of the Newton–Raphson solution technique.

1.1 A simple non-linear problem

The simple two-element truss model shown in Fig. 1.1 has often been used to illustrate some basic features of geometric non-linear behavior, see e.g. Bathe (1996, p. 494) and Crisfield (1991, pp. 2–13). The structure consists of two identical truss elements, loaded with a vertical force f at the center and simply supported at the other ends. The vertical displacement at the center is called u. In the initial configuration the length of the bars is l_0.

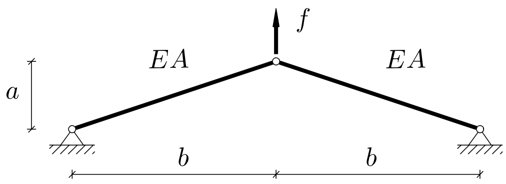

Fig. 1.1. Two-element truss model.

Application of the load leads to a deformed state with vertical displacement u of the central node, Fig. 1.2. The structure is assumed to be shallow, i.e. $a \ll b$. This permits series expansion of the square roots defining the original bar length l_0 and the bar length l corresponding to the current deformed state:

$$l_0 \;=\; \sqrt{b^2+a^2} \;\simeq\; b\Big(1+\frac{1}{2}\frac{a^2}{b^2}\Big), \tag{1.1}$$

$$l \;=\; \sqrt{b^2+(a+u)^2} \;\simeq\; b\left[1+\frac{1}{2}\Big(\frac{a+u}{b}\Big)^2\right]. \tag{1.2}$$

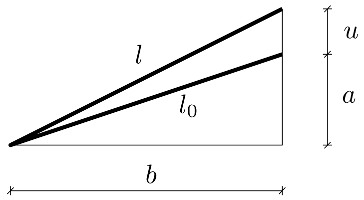

Fig. 1.2. Initial length l_0 and current length l.

The deformation of the bars is described by their elongation. A non-dimensional measure of deformation is the engineering strain, defined as the elongation relative to the original length,

$$\varepsilon \;=\; \frac{l-l_0}{l_0} \;\simeq\; \frac{a}{l_0}\frac{u}{l_0} \;+\; \frac{1}{2}\Big(\frac{u}{l_0}\Big)^2. \tag{1.3}$$

The first term is the linear part of the strain, while the second term is non-linear. A true measure of deformation must not be influenced by any rigid body motion of the bar, and thus a true deformation measure must be a non-linear function of the displacement component(s). If the displacement u is small relative to all characteristic lengths of the geometry – l_0 and a – the linear term will constitute a fair approximation, but if this approximation is used, some of the characteristic non-linear features of the problem are lost.

1.1.1 Equilibrium

The two bars are assumed to be linear elastic with axial stiffness EA, where E is the elastic modulus and A is the cross-section area. Thus, the axial force in each bar is expressed in terms of the strain as

$$N \;=\; EA\,\varepsilon \;\simeq\; EA\left[\frac{a}{l_0}\frac{u}{l_0} \;+\; \frac{1}{2}\Big(\frac{u}{l_0}\Big)^2\right]. \tag{1.4}$$

Equilibrium of the central node in the deformed state requires that the external force f is equal to the internal force $g(u)$ generated by deformation of the structure. Projection of the normal force gives

$$g(u) \;=\; 2\,N\,\frac{a+u}{l} \;\simeq\; \frac{2EA}{l_0^3}\big(au+\tfrac{1}{2}u^2\big)(a+u). \tag{1.5}$$

In non-dimensional form this is

$$g(u) = 2\,EA\left(\frac{a}{l_0}\right)^3\left[\frac{u}{a}+\frac{3}{2}\left(\frac{u}{a}\right)^2+\frac{1}{2}\left(\frac{u}{a}\right)^3\right], \tag{1.6}$$

where the normalized displacement is u/a. The load–displacement relation (1.6) is shown in Fig. 1.3 corresponding to a downward load.

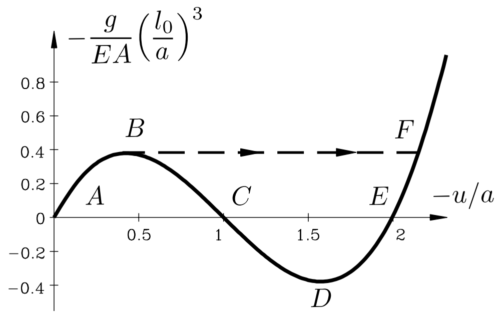

Fig. 1.3. Load–displacement curve for two-element truss.

From the unloaded state A an increasing downward load leads to a local maximum B. In this state the structure cannot support a further increase of the load. Thus, further increase of the load from B would lead to snap-through to F. The snap-through is an unstable dynamic process, and thus the load–displacement curve in Fig. 1.3 is not fully representative. Alternatively the structure may be loaded in displacement control, in which the central node is given a controlled downward displacement $-u$. This would require an increasing load from A to B, and then a decreasing load from B to C, where $u = -a$ and the two bars form a straight line. An upward force is now required to proceed to D and E, where the structure is stress-free, forming an angle symmetric to the original configuration with respect to the base line. Further downward load leads through F with increasing stiffness of the structure.

For a structure with one degree of freedom, the stiffness is a measure of the change in force for a given change in displacement. Thus, the tangent stiffness K is defined as the stiffness corresponding to infinitesimal changes in u and g:

$$K = \frac{dg}{du}. \tag{1.7}$$

In the present case the tangent stiffness K follows from straightforward differentiation of (1.6):

$$K = \frac{2EA}{l_0}\left(\frac{a}{l_0}\right)^2\left[1+3\left(\frac{u}{a}\right)+\frac{3}{2}\left(\frac{u}{a}\right)^2\right]. \tag{1.8}$$

Although this expression defines the tangent stiffness K, it does not convey the physics of the problem very clearly. This is better accomplished by differentiation of the equilibrium equation (1.5):

$$K = \frac{d}{du}\left(2N\frac{a+u}{l_0}\right) = 2\frac{EA}{l_0}\left(\frac{a+u}{l_0}\right)^2 + 2\frac{N}{l_0}. \tag{1.9}$$

Here $a+u$ is the height of the structure in the current state, while N is the current value of the axial force. The first term is due to changes in the normal force N, while the second term is due to changes in the geometric configuration with constant normal force N. Sometimes the first term is separated into a constant corresponding to $u=0$ and the rest, whereby (1.9) takes the form

$$\begin{aligned} K &= 2\frac{EA}{l_0}\left(\frac{a}{l_0}\right)^2 + 2\frac{EA}{l_0}\frac{2au+u^2}{l_0^2} + 2\frac{N}{l_0} \\ &= K_0 + K_u + K_\sigma, \end{aligned} \tag{1.10}$$

where K_0 is the linear stiffness, K_u is the initial displacement stiffness, and K_σ is the initial stress stiffness. In an incremental procedure, where the geometry is updated, the current value of u is absorbed in the updated value of a, and in that case the initial displacement stiffness K_u vanishes.

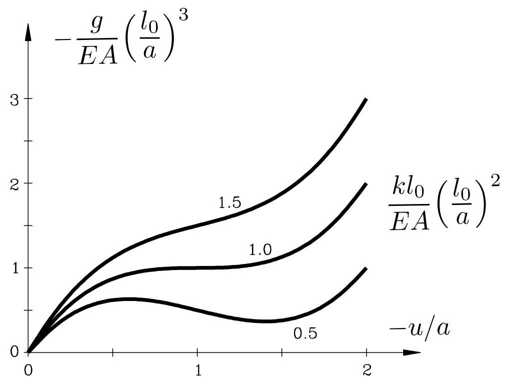

Fig. 1.4. Load–displacement curve for two-element truss with spring.

A family of load–displacement curves with different degrees of non-linearity can be obtained by introducing a vertical linear elastic spring with stiffness

k at the central node of the structure. The load–displacement relation (1.6) is changed to

$$g(u) = 2EA\Big(\frac{a}{l_0}\Big)^3\Big[\frac{u}{a}+\frac{3}{2}\Big(\frac{u}{a}\Big)^2+\frac{1}{2}\Big(\frac{u}{a}\Big)^3\Big] + ku \tag{1.11}$$

and the tangent stiffness (1.8) to

$$K = \frac{2EA}{l_0}\Big(\frac{a}{l_0}\Big)^2\Big[1+3\Big(\frac{u}{a}\Big)+\frac{3}{2}\Big(\frac{u}{a}\Big)^2\Big] + k. \tag{1.12}$$

Figure 1.4 shows the load–displacement curve for different values of the spring stiffness k. For $k \geq EAa^2/l_0^3$ the variation of load with displacement is monotonic, corresponding to $K \geq 0$.

1.1.2 Virtual work and potential energy

The load–displacement relations (1.6) and (1.11) were obtained from equilibrium of the center node. For structures with more degrees of freedom or more complicated elements it is often convenient to make use of the principle of virtual work. Essentially, the principle of virtual work is a restatement of a set of equilibrium equations, where each equation is multiplied by a corresponding infinitesimal virtual displacement component. With an appropriate definition of the force and displacement components summation of their products forms a scalar invariant, known as the virtual work.

In the particular example of the two-element truss with an elastic spring the equilibrium equation can be written as

$$2\,N\,\frac{a+u}{l} + ku - f = 0. \tag{1.13}$$

Multiplication by a virtual displacement δu gives the virtual work equation

$$\delta V = 2\,N\,\frac{a+u}{l}\,\delta u + (ku)\delta u - f\delta u = 0. \tag{1.14}$$

The displacement factor in the first term is similar to the first variation of the strain (1.3):

$$\delta\varepsilon = \frac{\partial}{\partial u}\Big[\frac{a}{l_0}\frac{u}{l_0}+\frac{1}{2}\Big(\frac{u}{l_0}\Big)^2\Big]\delta u = \frac{a+u}{l_0}\,\frac{\delta u}{l_0}. \tag{1.15}$$

If, for the time being, the difference between l_0 and l is neglected, the virtual work equation (1.14) can now be written as

$$\delta V \simeq 2\int_0^{l_0} N\,\delta\varepsilon\,ds + (ku)\,\delta u - f\,\delta u = 0. \tag{1.16}$$

The integral is the internal virtual work of the bar elements, the second term is the virtual work of the elastic spring, while the last term is the external virtual work.

Apart from the factor l_0/l that is somehow missing, the use of virtual work in the present case where $\delta\varepsilon$ is constant within the elements is almost trivial. However, for more general problems with more degrees of freedom and non-trivial displacement fields within the elements, the principle of virtual work is an important tool for establishing the balance equations of the discretized model. The question of the factor l_0/l is discussed in Chapter 2, where the theory of non-linear bar elements is discussed more rigorously. Here, the relation between virtual work and potential energy is discussed briefly before turning to elementary numerical solution methods for non-linear equilibrium equations.

When the internal forces such as the axial force N and the spring force ku are functions of the state of displacement given by u, and the external load is also a function of u, the virtual work δV can be considered as the differential of an energy function $\Phi(u)$ – the potential energy. In the present case (1.16) is written as

$$\delta\Phi(u) = 2\int_0^{l_0} EA\,\varepsilon\delta\varepsilon\, ds + ku\,\delta u - f\,\delta u. \tag{1.17}$$

This relation can be integrated with respect to the displacement u, giving the following expression for the potential energy:

$$\Phi(u) = 2\int_0^{l_0} \tfrac{1}{2}EA\varepsilon^2\, ds + \tfrac{1}{2}k\,u^2 - fu. \tag{1.18}$$

The potential energy is the internal strain energy of the structure, including the spring, minus the external work represented by fu. For linear elastic structures it may be simpler to derive the equilibrium equations from the potential energy by considering an incremental change δu of the displacements. However, the principle of virtual work is valid irrespective of the specific material behavior, and thus the principle of virtual work has become the method of choice for setting up equilibrium equations.

1.2 Simple non-linear solution methods

For a system with only one degree of freedom non-linear behavior can often be described explicitly as a function of the displacement u, and the problem may then be considered as one of displacement control. However, in the case of several degrees of freedom the use of displacement control is non-trivial, and most problems are formulated in terms of a load history, for which the corresponding displacement history is to be calculated. This requires the solution of a system of non-linear equations. Here some of the simpler methods for solving non-linear equations are briefly introduced, leaving more specialized techniques to Chapter 8. The methods are illustrated for a single degree of freedom and then generalized to matrix form.

1.2.1 Explicit incremental method

An explicit incremental method, often called the Euler explicit method, is obtained by replacing the differentials in the definition (1.7) of the tangent stiffness with finite increments Δf and Δu:

$$\Delta u \;=\; K^{-1}\,\Delta f. \tag{1.19}$$

The load–displacement history is described by a number of increments Δf_n, Δu_n, $n = 1, 2, \ldots$ defining the states

$$f_n \;=\; f_{n-1} + \Delta f_n, \quad u_n \;=\; u_{n-1} + \Delta u_n, \quad n = 1, 2, \ldots \tag{1.20}$$

In the explicit incremental method the tangent stiffness K corresponds to the state at the beginning of the increment. Thus, the precise form of (1.19) is

$$\Delta u_n \;=\; K^{-1}(u_{n-1})\,\Delta f_n, \quad n = 1, 2, \ldots \tag{1.21}$$

This procedure is illustrated in Fig. 1.5.

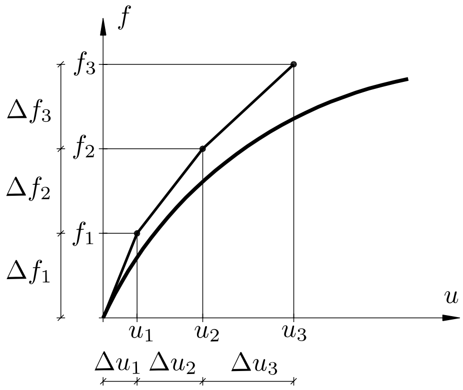

Fig. 1.5. Explicit incremental method.

It is seen that the computed states deviate more and more from the exact load–displacement curve. There are two reasons for this: the tangent stiffness of each increment is taken at the left end-point and in this particular case overestimates the stiffness, and deviations from the exact curve are added to a cumulative error. While it is difficult to use an exact representation for the stiffness corresponding to the full increment, the problem of increasing deviations can be countered by introducing equilibrium iterations as discussed in the following.

The explicit incremental method is easily generalized to multi-degree of freedom systems. Let the displacement vector be $\mathbf{u}$ and the corresponding load vector $\mathbf{f}$. The tangent stiffness matrix $\mathbf{K}$ is then defined by

$$d\mathbf{f} \;=\; \mathbf{K}(\mathbf{u})\, d\mathbf{u}. \tag{1.22}$$

The corresponding explicit incremental method is

$$\Delta\mathbf{u}_n \;=\; \mathbf{K}^{-1}(\mathbf{u}_{n-1})\,\Delta\mathbf{f}_n, \quad n = 1, 2, \ldots \tag{1.23}$$

The use of the inverse matrix $\mathbf{K}^{-1}$ in (1.23) should not be taken literally. In practice the matrix $\mathbf{K}$ is factored and the product $\mathbf{K}^{-1}\Delta\mathbf{f}$ found by back substitution.

1.2.2 Newton–Raphson method

In order to avoid accumulating errors in each additional load step, equilibrium iterations may be used to establish equilibrium to a desired degree of accuracy at each load step. This procedure is a special instance of the Newton–Raphson method, well known from numerical analysis. In principle, the method works by applying two steps intermittently: (i) check if equilibrium is satisfied to within the desired accuracy; (ii) if not, make a suitable adjustment of the state of deformation.

The first step consists in checking the equilibrium equation. This is done by forming the difference between the external load f and internal force $g(u)$,

$$r(u, f) \;=\; f \;-\; g(u) \;=\; 0, \tag{1.24}$$

where $r(u, f)$ is called the residual force. In a state of equilibrium the internal force $g(u)$ is equal to the external load f, and thus the residual vanishes. In practice, lack of equilibrium will be produced at the beginning of each load increment, where the load f is increased, while no new displacement estimate u is yet available. Thus, the need arises for obtaining an improved estimate of the state of displacement u.

In the absence of equilibrium an improved estimate of the displacement u is obtained from a linearized form of the residual $r(u + \delta u, f)$ around the known residual $r(u, f)$,

$$r(u + \delta u, f) \;=\; r(u, f) \;+\; \delta r(u, f) \;+ \cdots \;=\; 0. \tag{1.25}$$

The dots indicate higher-order terms, because δr is only a linearized form of the increment of the residual. In the classic form of equilibrium iterations the load f is assumed fixed within the given load step, and thus the increment of the residual only depends on the internal force $g(u)$. The linearized increment is then given by the first derivative of the internal force as

$$\delta r = -\frac{dg(u)}{du}\,\delta u = -K(u)\,\delta u. \tag{1.26}$$

Here the tangent stiffness K, introduced in (1.7), has been introduced. The displacement increment is now obtained from the linearized form of (1.25) by substitution of the tangent stiffness relation (1.26). When rearranging the terms in (1.25), the linearized equation becomes

$$K(u)\,\delta u = r. \tag{1.27}$$

In this equation the residual $r(u, f)$ is known, as it relates to the current state of load f and displacement u. The tangent stiffness $K(u)$ at the current displacement state u can also be calculated. Thus, this equation permits determination of the displacement increment δu,

$$\delta u = K^{-1}(u)\,r. \tag{1.28}$$

Once the displacement increment δu is determined, the current displacement state is updated to

$$u^i = u^{i-1} + \delta u^i. \tag{1.29}$$

In this equation the superscript is used to indicate that the iteration i changes the estimated displacement from u^{i-1} to u^i. In a computer program the iteration superscript i is not needed, as the register containing u^{i-1} is simply overwritten by the new value u^i according to the assignment statement

$$u := u + \delta u. \tag{1.30}$$

Here, $:=$ is the assignment operator, implying that the variable u is assigned a new value. In this book many of the algorithms are presented in the form of pseudocode – i.e. a code format that appears like high-level programs such as MATLAB. In the pseudocode presented here assignments are indicated by the normal equality sign, as all equalities are assignment statements.

The Newton–Raphson equilibrium iteration procedure is illustrated in Fig. 1.6. The figure shows load step n. This load step starts from a state of equilibrium already established at the previous load f_{n-1} with displacement u_{n-1}. The load step is initiated by increasing the load by Δf_n to f_n. This generates the first residual $r_n^1 = \Delta f_n$. This residual and the tangent stiffness