

Computation and Modelling in Insurance and Finance

Scientific computing is as critical for the analysis of risk in insurance and finance as are mathematics and statistics, and it should be taught jointly with them. This book offers such an integrated approach at an introductory level and provides readers with much of what is needed in practice, including how simulation programs are designed, used and reused (with modifications) as situations change. Complex problems with risk from many sources are discussed, as is the sensitivity of conclusions on assumptions and historical data.

The tools of modelling and simulation are outlined in Part I with special emphasis on the Monte Carlo method and its use. Part II deals with general insurance and Part III with life insurance and financial risk. Algorithms that can be implemented on any programming platform are spread throughout, and a program library written in R is included. Numerous figures and experiments with R code illustrate the text.

The author's non-technical approach is ideal for graduate students, the only prerequisites being introductory courses in calculus and linear algebra, probability and statistics. The book will also be useful for actuaries and other analysts in the industry looking to update their skills.

ERIK BØLVIKEN, with broad experience as an applied statistician, holds the Chair of Actuarial Science at the University of Oslo and was for many years a partner in Gabler and Partners, Oslo.

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Preface

The book is organized as a broad introduction to concepts, models and computational techniques in Part I and with general insurance and life insurance/financial risk in Parts II and III. The latter are largely self-contained and can probably be read on their own. Each part may be used as a basis for a university course; we do that in Oslo. Computation is more strongly emphasized than in traditional textbooks. Stochastic models are defined in the way they are simulated in the computer and examined through numerical experiments. This cuts down on the mathematics and enables students to reach ‘advanced’ models quickly. Numerical experimentation is also a way to illustrate risk concepts and to indicate the impact of assumptions that are often somewhat arbitrary. One of the aims of this book is to teach how the computer is put to work effectively.

Other issues are error in risk assessments and the use of historical data, each of which are tasks for statistics. Many of the models and distributions are presented with simple fitting procedures, and there is an entire chapter on error analysis and on the difference between risk under the underlying, real model and the one we actually use. Such error is in my opinion often treated too lightly: we should be very much aware of the distinction between the complex, random mechanisms in real life and our simplified model versions with deviating parameters. In a nebulous and ever-changing world modelling should be kept simple and limited to the essential.

The reader must be familiar with elementary calculus, probability and matrix algebra (the last two being reviewed in appendices) and should preferably have some programming experience. These apart, the book is self-contained with concepts and methods developed from scratch. The text is equipped with algorithms written in pseudo-code that can be programmed on any platform whereas the exercises make use of the open-source R software which permits Monte Carlo simulation of quite complex problems through a handful of commands. People can teach themselves the tricks of R programming by following the instructions in the exercises (my recommendation), but it is also possible to use the associated R library

passively. The exercises vary from the theoretical to the numerical. There is a good deal of experimentation and model comparison.

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Erik Bølviken
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