Introduction

This is a book primarily about symmetrization of real-valued functions and sets. Many extremal problems in mathematics and physics have symmetric solutions, the quintessential example being the isoperimetric inequality (see Chapter 4) that among all sets with given volume, the ball possesses minimal surface area. The book develops and applies symmetrization techniques for problems in geometry, partial differential equations, and complex analysis.

Other treatments of symmetrization with applications to analysis and partial differential equations can be found in the works of Bandle (1980), Bennett and Sharpley (1988), Kawohl (1985), Kesavan (2006), Lieb and Loss (1997), and Pólya and Szegő (1951). For applications to complex analysis see Duren (1983), Hayman and Kennedy (1976), Hayman (1989, 1994), and Dubinin (2014). For applications to Fourier analysis and hyperbolic geometry, one may consult Beckner (1995).

Each chapter ends with Notes that contain historical remarks and additional information.

Chapter 1 presents the theory of rearrangements of functions, where one compares a real-valued function f on a measure space $(X, \mathcal{M}, \mu)$ with another function g, defined on a possibly different measure space, such that f and g have the same "size." The notion of size corresponds to the distribution function $\lambda_f(t) = \mu(f > t)$. To avoid technical difficulties with infinity, we always assume that $\lambda_f(t) < \infty$, for every $t > \operatorname{ess\,inf} f$. We consider f and g to have the same size if they have the same distribution function, in which case f and g are called rearrangements of each other. We would like to find a rearrangement g that has "more symmetry" than f.

The simplest case (§1.2) is the decreasing rearrangement of f, denoted f^*, which is a decreasing one-variable function defined on the interval $[0, \mu(X)]$.

Next in simplicity is the symmetric decreasing rearrangement on $\mathbb{R}^n$ (§1.6), written $f^\#(x)$. It has the property that $(f^\# > t)$ is a ball centered at the origin. Before studying $f^\#$ prerequisites in measure theory are covered (§§1.3–1.4) in order to present a general version of Ryff's factorization theorem (1970). Ryff's theorem asserts that if $(X, \mathcal{M}, \mu)$ is a nonatomic measure space with $\mu(X) < \infty$ and $f\colon X \to \mathbb{R}$ is $\mathcal{M}$ measurable, then a measure preserving transformation $T\colon X \to [0, \mu(X)]$ exists such that $f = f^* \circ T$ for almost every $x \in X$. Note that if T is measure preserving, then f and $f \circ T$ have the same distribution function. A particular case is when $T(x) = \alpha_n |x|^n$, with α_n the volume of the unit ball in $\mathbb{R}^n$. In that case $f^\# = f^* \circ T$ (see §1.6), which connects the symmetric decreasing rearrangement to the decreasing rearrangement through the change of variable T.

Another type of rearrangement central to this book is the polarization of f with respect to an affine hyperplane $H \subset \mathbb{R}^n$, denoted by f_H (§1.7). Polarization involves moving the larger values of f preferentially to one side of the hyperplane. Polarization with respect to all hyperplanes that do not contain the origin yields the symmetric decreasing rearrangement $f^\#$.

The chapter ends with convergence theorems for f^* and $f^\#$, covering the cases of almost everywhere convergence and convergence in measure.

Examples and graphs are included throughout the chapter, in line with the author's pedagogical intentions. Some new notions are introduced first in the discrete case, where functions are just finite sequences and all calculations can be carried out explicitly.

Chapter 2 covers the foundational inequalities for integrals of functions on $\mathbb{R}^n$. In Baernstein's approach a key notion is that of an AL function $\Psi(x, y)$, which generalizes the condition of nonnegative mixed partials $\Psi_{xy} \geq 0$. The two key results in this chapter are that symmetric decreasing rearrangement of a continuous function decreases its modulus of continuity, and that certain integral expressions increase when functions are replaced by their symmetric decreasing rearrangements.

The proof presented for the decrease of modulus of continuity (Theorem 2.12) is based on elementary polarization inequalities and the Arzelà–Ascoli theorem, and does not rely on other inequalities such as the isoperimetric or Brunn–Minkowski type inequalities.

Given nonnegative functions f, g, along with a nonnegative kernel K and an AL function $\Psi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, the basic inequality in Theorem 2.15 says that a certain integral expression increases under symmetrization:

$$\int_{\mathbb{R}^{2n}} \Psi(f(x), g(y)) K(|x-y|)\, dx\, dy \leq \int_{\mathbb{R}^{2n}} \Psi(f^\#(x), g^\#(y)) K(|x-y|)\, dx\, dy.$$

The proof is presented in stages. First, an analogous inequality is proved in the simple case of a space consisting of two points (Theorem 2.8), and then for the case of polarization with respect to an affine subspace (Theorem 2.9), and finally for the symmetric decreasing rearrangement (Theorem 2.15). This structured approach permits easy modification later (Chapter 7) to spheres and hyperbolic spaces. The proof is done first in the case of continuous Ψ, which is the most important case for applications, and completed in §§2.8–2.9 for general AL functions.

In §2.7 many direct consequences of Theorem 2.15 are presented, including the classical Hardy–Littlewood inequality

$$\int_{\mathbb{R}^n} fg\,dx \le \int_{\mathbb{R}^n} f^\# g^\#\,dx, = \int_{\mathbb{R}^+} f^* g^*\,dx,$$

as well as the contractivity of rearrangement in the L^∞-norm (Corollary 2.23).

Chapter 3 develops the basic Dirichlet integral inequalities for symmetric decreasing rearrangement. The main result is the inequality

$$\int_{\mathbb{R}^n} |\nabla f^\#|^p\,dx \le \int_{\mathbb{R}^n} |\nabla f|^p\,dx, \quad 1 \le p < \infty,$$

for $f \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R})$ satisfying $\lambda_f(t) < \infty$ for all $t > \inf f$ (Theorem 3.7) and its extension to $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^+)$ (Theorem 3.20). The inequality when $p = \infty$ is easier, $\|\nabla f^\#\|_{L^\infty(\mathbb{R}^n)} \le \|\nabla f\|_{L^\infty(\mathbb{R}^n)}$, and follows from the monotonicity of the modulus of continuity (Theorem 3.6). Background on Lipschitz functions is given in §3.1. The proof of Theorem 3.7 (the Lipschitz case) is in §3.2, ultimately based on the basic inequality in Theorem 2.15. Various comments are made on the equality case. This section also includes a version valid for nonnegative functions on a domain $\Omega \subset \mathbb{R}^n$ (Corollary 3.9), assuming the function vanishes on the boundary.

Section 3.3 presents a more general inequality for Φ-Dirichlet integrals

$$\int_{\mathbb{R}^n} \Phi(|\nabla f^\#|)\,dx \le \int_{\mathbb{R}^n} \Phi(|\nabla f|)\,dx,$$

where $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is convex and increasing with $\Phi(0) = 0$. The proof is again based on Theorem 2.15. Another approach due to Dubinin based on polarization is included too.

Sections 3.4 and 3.5 include background material on Sobolev spaces and functional analysis needed to extend the Dirichlet integral inequality to functions in the Sobolev space $W^{1,p}(\mathbb{R}^n, \mathbb{R}^+)$. The extension is presented in §3.6. The chapter ends with §3.7, discussing the continuity of the rearrangement operator $f \mapsto f^\#$ in various situations. The operator is continuous in $L^p(\mathbb{R}^n, \mathbb{R}^+)$, continuous at the zero function in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^+)$, and continuous everywhere in $W^{1,p}(\mathbb{R}, \mathbb{R}^+)$ (dimension $n = 1$), but is discontinuous at a general Sobolev function when $n \ge 2$. The condition for continuity at f, the coarea regularity condition discovered by Almgren and Lieb, is presented in this section.

Chapter 4 is devoted to the isoperimetric inequality and sharp Sobolev inequalities. It begins with a review of geometric measure theory tools (Hausdorff measures, area formula, and Gauss–Green theorem) used in this and later chapters. The convention of Evans and Gariepy (1992) is followed in this chapter: "measure" means "outer measure."

Three isoperimetric inequalities are presented: for perimeters (Theorem 4.10), for Hausdorff measures (Corollary 4.13), and for Minkowski content (Theorem 4.16). If $E \subset \mathbb{R}^n$ with finite perimeter, finite measure, or finite Minkowski content, one has

$$\begin{aligned} P(E) &\geq P(E^\#), \\ \mathcal{H}^{n-1}(\partial E) &\geq \mathcal{H}^{n-1}(\partial(E^\#)), \\ \mathcal{M}_*^{n-1}(\partial E) &\geq \mathcal{M}^{n-1}(\partial(E^\#)), \end{aligned}$$

where $P(E^\#) = \mathcal{H}^{n-1}(\partial(E^\#)) = \mathcal{M}^{n-1}(\partial(E^\#)) = n\alpha_n^{1/n}\mathcal{L}^n(E)^{\frac{n-1}{n}}$, and $E^\#$ is a ball of the same volume as E. (Here $\mathcal{L}^n$ is the n-dimensional Lebesgue measure.) All three isoperimetric inequalities are deduced from the fact that symmetrization decreases the Dirichlet integral (Theorem 3.7) or the variation of a function (Theorem 4.8).

Additional facts from geometric measure theory (the coarea formula and polar coordinates) are stated in §4.5. This section also shows that the coarea formula and the isoperimetric inequality for perimeter together imply decrease of the Dirichlet integral under symmetrization.

Section 4.6 presents the proof of the sharp Sobolev embedding inequalities for $f \in BV(\mathbb{R}^n)$, $n \geq 2$, which is

$$\|f\|_{\frac{n}{n-1}} \leq n^{-1}\alpha_n^{-1/n}V(f).$$

Equality holds when $f = \chi_B$ for some ball $B \subset \mathbb{R}^n$. The proof is reduced to the radial case by symmetrization. Another proof based on the isoperimetric inequality and the coarea formula is also included. This shows that the sharp Sobolev inequality in $BV(\mathbb{R}^n)$ is indeed equivalent to the sharp isoperimetric inequality. Section 4.7 gives the corresponding sharp result for $W^{1,p}(\mathbb{R}^n)$ when $1 < p < n$, $n \geq 2$:

$$\|f\|_{p^*} \leq (n\alpha_n^{1/n})^{-1}(p^*/p')^{1/p'}\left(\frac{p'}{n}\frac{\Gamma(n)}{\Gamma(n/p)\Gamma(n/p')}\right)^{1/n}\|\nabla f\|_p,$$

where $p^* = np/(n-p)$ is the Sobolev conjugate of p, and $p' = p/(p-1)$ is the Hölder conjugate, and Γ is the Gamma function. Equality holds for $g_{n,p}(x) = (1 + |x|^{p'})^{-n/p^*}$. The proof of this inequality starts with a symmetrization to reduce to radial functions, and then follows a constructive version of the strategy of the proof by Cordero-Erausquin, Nazaret and Villani (2004) based on Monge–Kantorovich mass transportation ideas. The point is that in this proof, the transport map is explicitly constructed.

The last part of the chapter, §4.8, deals with the cases $p = n$ (Moser's theorem) and $p > n$ (Morrey's embedding theorem). Sharp inequalities are not known in the latter case, while partial results are available in the former.

Chapter 5 covers three classical topics in symmetrization, and includes historical remarks as well as the needed background in physics to guide the reader. The first result is that symmetrizing a fixed membrane into a disk of the same area decreases its principal frequency (the first eigenvalue of the Laplacian with Dirichlet boundary conditions), as conjectured by Rayleigh in 1877 and proved independently by Faber (1923) and Krahn (1925). The second result is that symmetrization increases the torsional rigidity of a planar domain, as conjectured by St Venant in 1856 and proved by Pólya (1948). Lastly, a closed ball in $\mathbb{R}^3$ is shown to have the smallest Newtonian capacity among all compact sets with the same volume. This conjecture was raised by Poincaré in 1887 and proved by Szegő (1930). The proofs depend on the decrease of the Dirichlet integral under symmetric decreasing rearrangement of the function.

Background on weak solutions and spectral theory for the Laplace operator is presented in §§5.1–5.2 with all details carefully presented. In §5.3 we reach the proof of the Faber–Krahn theorem: when Ω is a bounded open set in $\mathbb{R}^n$ and $\Omega^\#$ is a ball of the same volume, the first eigenvalue λ_1 of the Laplacian is smallest for the ball: $\lambda_1(\Omega) \geq \lambda_1(\Omega^\#)$. The proof relies on expressing the first eigenvalue as the minimum value of the Rayleigh quotient, by

$$\lambda_1(\Omega) = \min_u \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2\, dx},$$

where the minimum is over all $u \in W_0^{1,2}(\Omega)$ with $u \not\equiv 0$.

Two useful domain approximation lemmas are proved in §5.4, and then the Newtonian capacity of a compact set is developed from Coulomb's inverse square law in electrostatics, in §5.5. Szegő's Theorem's follows from the variational characterization of Newtonian capacity in terms of Dirichlet integrals:

$$\mathrm{Cap}(K) = \inf \left\{ \frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla v|^2\, dx \colon v \in \mathcal{A}(K) \right\}$$

where the class of admissible functions is

$$\mathcal{A}(K) = \{v \in \mathrm{Lip}(\mathbb{R}^3)\colon\ 0 \leq v \leq 1 \text{ in } \mathbb{R}^3,\ v = 1 \text{ on } K,\ \lim_{|x|\to\infty} v(x) = 0\}.$$

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The key point is that if v is admissible for K then $v^\#$ is admissible for the symmetrized set $K^\#$. Extensions to variational p-capacities, Riesz α-capacities, and logarithmic capacities are considered in §5.6.

The torsional rigidity of a bounded open set $\Omega \subset \mathbb{R}^n$ is the quantity

$$T(\Omega) = 2 \int_\Omega u(x)\, dx,$$

where u satisfies $\Delta u = -2$ in Ω with $u = 0$ in $\partial\Omega$. It turns out that $u(x)$ can also be interpreted as the expected lifetime of Brownian motion starting at $x \in \Omega$ and that $T(\Omega)/2|\Omega|$ equals the average lifetime of a particle born somewhere in Ω. The key result of §5.7 is that symmetrization increases both quantities, that is,

$$T(\Omega) \leq T(\Omega^\#).$$

Chapter 6 discusses Steiner symmetrization. The Steiner symmetrization of a set or function on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$ is obtained by performing symmetric decreasing rearrangement on the k-dimensional slice $\mathbb{R}^k \times \{z\}$, for each $z \in \mathbb{R}^m$.

Basic properties of symmetric decreasing rearrangement that were developed in Chapter 1 are adapted to Steiner symmetrization in §6.2, and properties of polarization are adapted in §6.3. Then Theorem 6.8 is an analogue of the main inequality (Theorem 2.15), taking the form

$$\int \Psi(f(x), g(\bar{x}))K(|x - \bar{x}|)\, dx\, d\bar{x} \leq \int \Psi(f^\#(x), g^\#(\bar{x}))K(|x - \bar{x}|)\, dx\, d\bar{x}.$$

In §6.5 we see Steiner symmetrization decreases the modulus of continuity (Theorem 6.10) and the diameter (Theorem 6.12), and acts contractively on $L^\infty(X)$ (Theorem 6.14).

When considering the effect of Steiner symmetrization on Dirichlet integrals (§6.6), one first splits the gradient as

$$\nabla f(x) = \big(\nabla_y f(x), \nabla_z f(x)\big)$$

where $x = (y, z)$. Applying on each slice the result for symmetric decreasing rearrangement from Chapter 3, we find under suitable conditions on f that

$$\int \Phi(|\nabla_y f^\#(y, z)|)\, dy \leq \int \Phi(|\nabla_y f(y, z)|)\, dy$$

for each z. Integrating over z gives

$$\int \Phi(|\nabla_y f^\#(x)|)\, dx \leq \int \Phi(|\nabla_y f(x)|)\, dx.$$

The corresponding inequalities for the transverse gradient $\nabla_z f$ and full gradient ∇f are obtained in Theorem 6.16:

$$\int \Phi(|\nabla_z f^{\#}(y,z)|)\,dy \le \int \Phi(|\nabla_z f(y,z)|)\,dy$$

and

$$\int \Phi(|\nabla f^{\#}(y,z)|)\,dy \le \int \Phi(|\nabla f(y,z)|)\,dy$$

for each z. Once again, integrating over z yields inequalities on all of $\mathbb{R}^n$.

While the above statements are simple, the proofs requires a technical lemma postponed to §6.7. In §6.8 the case of p-Dirichlet integrals is considered for Sobolev functions (Theorem 6.19). The case $p > 1$ follows from Theorem 6.16, but the case $p = 1$ needs additional work.

Steiner symmetrization decreases perimeter and Minkowski content, but in general it is not known whether it decreases the $(n-1)$-dimensional Hausdorff measure (§6.9). Steiner symmetrization also decreases the principal frequency and various capacities, and increases the torsional rigidity and mean lifetime of a Brownian particle (§6.10).

Chapter 7 covers symmetrization in the sphere $\mathbb{S}^n$, hyperbolic space $\mathbb{H}^n$, and Gauss space, and includes as an application a landmark theorem of Gehring on quasiconformal mappings.

Spheres and hyperbolic spaces have a canonical distance and measure, and possess rich isometry groups of measure preserving mappings. There are plenty of hyperplanes in which to polarize, and so most of the theory from Chapters 2–6 can be extended.

Sections 7.1 and 7.2 introduce the distance and measure on the sphere. The distance $d(x,y)$ is the length of the shortest circular arc joining points x and y, and so $0 \le d(x,y) \le \pi$. The measure σ_n is the restriction of the n-dimensional Hausdorff measure $\mathcal{H}^n$ to $\mathbb{S}^n$. The unit vector e_1 plays the role of origin, in the sphere, and the metric balls centered at this origin are the open spherical caps

$$K(\theta) = \{x \in \mathbb{S}^n : d(x,e_1) < \theta\}, \qquad \theta \le \pi.$$

Hyperplanes in $\mathbb{S}^n$ are given by the intersection of the sphere with hyperplanes in $\mathbb{R}^{n+1}$ that pass through the origin. Hence the polarization theory from §1.7 carries over to the sphere. Symmetric decreasing rearrangement for sets and functions extends to the sphere also, using spherical caps rather than Euclidean balls.

Spherical analogs of inequalities from Chapters 1 and 2 are developed in §7.3. The basic polarization inequality is Theorem 7.2, and the foundational inequality for integrals of functions on $\mathbb{S}^n$ under symmetric decreasing rearrangement is Theorem 7.3. The proofs are somewhat simpler than in Euclidean space, due to compactness of the sphere. In §7.4 one finds the decrease of spherical Dirichlet integrals under symmetric decreasing rearrangement on the sphere (Theorem 7.4) and the spherical isoperimetric inequality for Minkowski content (Theorem 7.5).

Cap symmetrization on $\mathbb{R}^n$ is presented in §7.5, where spherical (k,n)-cap symmetrization corresponds to (k,n)-Steiner symmetrization except now rearranging on k-spheres rather than k-planes. For example, circular symmetrization in the complex plane is exactly $(1,2)$-cap symmetrization, with the function made symmetric decreasing about the positive real axis, on each circle centered at the origin.

Section 7.6 is devoted to symmetrization in the hyperbolic space $\mathbb{H}^n$, which is modeled by the unit ball $\mathbb{B}^n$ endowed with the hyperbolic metric

$$ds = \frac{2}{1-|x|^2}\,|dx|,$$

where $|dx|$ is the Euclidean length element. The corresponding hyperbolic measure has density $2^n(1-|x|^2)^{-n}$. Polarization is defined in terms of hyperbolic hyperplanes, and hyperbolic symmetric decreasing rearrangement is constructed in terms of balls centered at the origin, but with respect to the hyperbolic measure rather than Euclidean measure. The majority of the symmetrization results from Chapters 1–5 are shown to hold for hyperbolic symmetric decreasing rearrangement.

Section 7.7 presents a brief discussion of symmetrization in the Gauss space $(\mathbb{R}^n, d\mu)$, where $d\mu = (2\pi)^{-n/2}e^{-|x|^2/2}\,dx$. Here sets and functions are rearranged with respect to the measure $d\mu$. The lack of appropriate hyperplanes makes the theory quite different from Euclidean, spherical, or hyperbolic symmetrization. A version of the isoperimetric inequality for the Gaussian Minkowski content can be proved by using the fact that Gauss space is the limit of spheres of increasing radius and dimension going to infinity; see Corollary 7.12.

In the final section, §7.8, the basic theory of quasiconformal mappings in $\mathbb{R}^n$ is discussed, including the equivalence of the analytic and geometric definitions of quasiconformality (Theorem 7.15). The sharp Hölder continuity exponent $1/K$ for K-quasiconformal mappings is obtained by using $(n-1,n)$-cap symmetrization, in Theorem 7.16 and Corollary 7.17. This is a celebrated theorem of Gehring (1962).

Chapter 8 studies symmetrization and convolution. The Riesz–Sobolev convolution theorem for nonnegative functions f, g, h on $\mathbb{R}^n$ asserts that the triple convolution

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(-x)g(y)h(x-y)\,dxdy = f*g*h(0)$$

increases when f, g, h are replaced by their symmetric decreasing rearrangements. The theorem is proved for functions on the circle $\mathbb{S}^1$ in §8.1, using ideas suggested by the star function in Chapter 9. The version on the circle implies the version on the real line (§8.2), which in turn implies the version in $\mathbb{R}^n$ (§8.3) for symmetric decreasing rearrangement and (k,n)-Steiner symmetrization. The Brunn–Minkowski inequality is proved in §8.4 as an application of Riesz–Sobolev.

A significant extension of the Riesz–Sobolev inequality, valid for multiple integrals with arbitrarily many functions, is the Brascamp–Lieb–Luttinger inequality proved in §8.5. It implies that the Dirichlet heat kernel increases under symmetrization (§8.6). On a bounded open set $\Omega \subset \mathbb{R}^n$ the Dirichlet Laplacian has eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots$, and by writing the heat kernel $K(x,y,t)$ as an eigenfunction series one arrives at the heat trace

$$\mathrm{Tr}(t,\Omega) = \int_\Omega K(x,x,t)\,dx = \sum_{j=1}^{\infty} e^{-\lambda_j t}.$$

Luttinger's Theorem 8.9 says the heat trace increases under rearrangement:

$$\mathrm{Tr}(t,\Omega) \le \mathrm{Tr}(t,\Omega^\#),$$

where $\Omega^\#$ denotes the symmetric decreasing rearrangement or (k,n)-Steiner symmetrization of the domain.

The Hardy–Littlewood–Sobolev inequality

$$\int_{\mathbb{R}^{2n}} f(x)g(y)|x-y|^{-\lambda}\,dxdy \le C\|f\|_p\|g\|_q$$

holds when $p > 1, q > 1, 0 < \lambda < n$. Section 8.7 presents a result of Lieb that determines the sharp constants for certain special values of the parameters. A key ingredient is to observe the conformal invariance of the integral (Proposition 8.12). Theorem 8.15 presents the sharp version (best constant) of the Hardy–Littlewood–Sobolev inequality for $1 < p < 2$ and $\lambda = 2n/p'$. In this case the extremals are constant multiples of $(a^2 + |x-v|^2)^{-n/p}$, where $a > 0$ and $v \in \mathbb{R}^n$.

In §8.8 and §8.10 the endpoint cases $\lambda \to n$ and $\lambda \to 0$ are investigated, following ideas of Beckner. The first case yields Gross's logarithmic Sobolev inequality (8.62, 8.63), as an infinite dimensional version of Beckner's logarithmic Sobolev inequality in $\mathbb{S}^n$ (Theorem 8.17). The second case gives sharp inequalities for exponential integrals known as the Lebedev–Milin inequality (8.69) and Onofri's inequality (8.70). In §8.9 Beckner's logarithmic Sobolev inequality is used to establish the hypercontractivity of the Poisson semigroup in a sharp range.

Chapter 9 marks the debut of the star function in the book. Each type of rearrangement $u^{\#}$ has an associated star function $u^{\star}$, which is an indefinite integral of $u^{\#}$. This chapter proves "subharmonicity" theorems for the star function, expressing the fact that if u satisfies a Poisson-type partial differential equation then $u^{\star}$ satisfies a related differential inequality. In the simplest case of a function u in the plane, subject to circular symmetrization, the result says that if u is subharmonic then so is $u^{\star}/r$. Subharmonicity yields comparison theorems for solutions of partial differential equations (Chapter 10), and extremal results in complex analysis (Chapter 11). Recall the complex plane with circular symmetrization is where the star function first made an impact.

Section 9.1 defines the star function in terms of the decreasing rearrangement on a general measure space, by

$$u^{\star}(x) = \int_0^x u^*(s)\,ds = \sup\left\{\int_E u\,d\mu :\ \mu(E) = x\right\}.$$

This formula motivates the star function definition for each of the specific geometries considered later in the chapter: spherical shells, spheres, Euclidean domains, and so on. Section 9.2 provides an overview of the chapter, and the next section establishes some facts on measurability.

The Laplacian is usually regarded as a differential operator, but it is more convenient in §9.4 to formulate the Laplacian as a limit of integral operators, so that later we can apply rearrangement results for convolutions. Specifically, the Laplacian at a point equals the difference between the average value of the function over a small neighborhood and its value actually at the point, as made precise by Lemma 9.5 for functions and Lemma 9.6 for measures.

The theory of the star function is easiest to grasp in the case of $(n-1,n)$-cap symmetrization on a spherical shell, because no boundary conditions need be imposed. Accordingly, we start with that case in §9.5. Given a measure with decomposition

$$d\mu = f\,d\mathcal{L}^n + d\tau - d\eta$$

where the function f is locally integrable, $\mathcal{L}^n$ is Lebesgue measure and τ and η are nonnegative measures, the cap symmetrization of μ is defined by

$$d\mu^{\#} = f^{\#} d\mathcal{L}^n + d\tau^{\#} - d\eta_{\#}.$$

Here $f^{\#}$ is symmetric decreasing on each sphere centered at the origin, $\tau^{\#}$ is the measure obtained by sweeping the mass of τ on each sphere to the positive x_1-axis, and $\eta^{\#}$ is obtained by spherically sweeping the mass of η