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Part I

Basic tools

1

Rate equations

Modeling lasers may be realized with different levels of sophistication. Rigorously it requires a full quantum treatment but many laser dynamical properties may be captured by semiclassical or even purely classical approaches. In this book we deliberately chose the simplest point of view, i.e. purely classical equations, and try to extract analytically as much information as possible. The basic framework of our approach is provided by the *rate equations*.

In their simplest version, they apply to an idealized active system consisting of only two energy levels coupled to a reservoir. They were introduced as soon as the laser was discovered to explain (regular or irregular, damped or undamped) intensity spikes commonly seen with solid state lasers (for a historical review, see the introduction in [18]). These rate equations are discussed and sometimes derived from a semiclassical theory in textbooks on lasers [19–22]. They capture the essential features of the response of a single-mode laser and they may be modified to account for specific effects such as the modulation of a parameter or optical feedback.

The most basic processes involved in laser operation are schematically represented in Figure 1.1. N_1 and N_2 denote the number of atoms in the ground and excited levels, respectively. The process of light–matter interaction is restricted to stimulated emission and absorption. This leads to the following rate equations for the number of laser photons n and the populations N_1 and N_2:

$$\frac{dn}{dT} = G(N_2 - N_1)n - \frac{n}{T_c}, \tag{1.1}$$

$$\frac{dN_2}{dT} = R_p - \frac{N_2}{T_1} - G(N_2 - N_1)n, \tag{1.2}$$

$$\frac{dN_1}{dT} = -\frac{N_1}{T_1} + G(N_2 - N_1)n. \tag{1.3}$$

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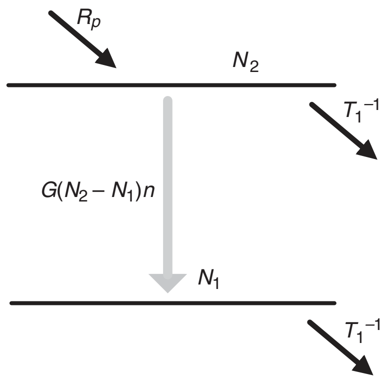

Fig. 1.1 Two-level system. R_p denotes the pumping rate, T_1^{-1} is the decay rate of the populations, and $G(N_2 - N_1)$ is the gain for stimulated emission.

In these equations, G is the gain coefficient for stimulated emission, T_c^{-1} is the decay rate due to the loss of photons by mirror transmission, scattering, etc., T_1^{-1} is the decay rate for each population, and R_p is the pumping rate. Introducing the population difference or population inversion $N \equiv N_2 - N_1$, Eqs. (1.1)–(1.3) reduce to the following two equations for n and N:

$$\frac{dn}{dT} = GNn - \frac{n}{T_c}, \tag{1.4}$$

$$\frac{dN}{dT} = -\frac{1}{T_1}(N - N_0) - 2GNn, \tag{1.5}$$

where $N_0 \equiv R_p T_1$ is the population difference in the absence of laser light. The decay rates T_c^{-1} and T_1^{-1} are identical to the parameters 2κ and $\gamma_{\parallel}$, respectively, in the "class B" laser equations [23, 6].

In practice, lasing action is realized with three or four energy level systems and the rate equations are more complicated (see Chapter 2). But for many lasers such as Nd^{3+}:YAG, CO_2, and semiconductor lasers, Eqs. (1.4) and (1.5) provide a good description of simple dynamical phenomena such as the laser relaxation oscillations or the build-up of laser radiation following either pump or loss switch. Supplemented by additional terms, these equations are also valid for the description of specific laser instabilities as we shall illustrate in the forthcoming chapters.

1.1 Dimensionless equations

Equations (1.4) and (1.5) depend on four physical parameters, namely, G, T_c, T_1, and N_0. In order to reduce the number of independent parameters, it is worthwhile

Table 1.1 *Characteristic times for common lasers.*

Laser	T_c(s)	T_1(s)	γ
CO_2	10^{-8}	4×10^{-6}	2.5×10^{-3}
solid state (Nd^{3+}:YAG)	10^{-6}	2.5×10^{-4}	4×10^{-3}
semiconductor (AsGa)	10^{-12}	10^{-9}	10^{-3}

to rewrite these equations in dimensionless form (for a dimensionless formulation, see, for example, [24]). Introducing new variables I, D, and t defined as

$$I \equiv 2GT_1 n, \quad D \equiv GT_c N, \quad \text{and} \quad t \equiv T/T_c \tag{1.6}$$

into Eqs. (1.4) and (1.5), we obtain the following equations for I and D (Exercises 1.8.1 and 1.8.4)

$$\frac{dI}{dt} = I(D-1), \tag{1.7}$$

$$\frac{dD}{dt} = \gamma(A - D(1+I)) \tag{1.8}$$

where A and γ are defined by

$$A \equiv GT_c N_0 \quad \text{and} \quad \gamma \equiv T_c/T_1. \tag{1.9}$$

Compared to the original equations (1.4) and (1.5), Eqs. (1.7) and (1.8) offer two clear advantages. First, we only have two independent parameters instead of the original four parameters. This means that Eqs. (1.7) and (1.8) are simpler to analyze or require fewer numerical simulations. Second, we may estimate these two parameters for different lasers, discover common ranges of values, and possibly propose approximations of the solution based on their respective values.

Table 1.1 gives the order of magnitude of T_c and T_1 for three common lasers. Although their ranges of values are quite different, we note that the ratio $\gamma \equiv T_c/T_1$ is typically a 10^{-3} small quantity. For microchip solid state lasers, γ may even reach 10^{-6} small values. A small γ is a key property of these lasers and, as we shall demonstrate, is responsible for their weak stability properties. On the other hand, A scales the pump in units of the pump at threshold and is typically in the range 1−10. It barely exceeds 10 in most common lasers although it may reach very high values in specific situations such as the "thresholdless laser" [25]. In addition to solid state lasers, earlier laser studies used He-Ne and Ar gas lasers. For the He-Ne and Ar gas lasers the value of γ is much larger than 1. Consequently, the evolution of the population inversion is very fast until the right hand side of Eq. (1.8) is zero. D then adiabatically follows the intensity as

$$D = \frac{A}{1+I} \tag{1.10}$$

and Eq. (1.7) reduces to

$$\frac{dI}{dt} = \left(\frac{A}{1+I} - 1\right) I. \tag{1.11}$$

Eq. (1.11) is a first order nonlinear equation. Lasers described by the single equation (1.11) are called "class A" lasers [23, 6]. Moreover, assuming $I < 1$, we may further simplify Eq. (1.11) by expanding $1/(1+I)$ and obtain

$$\frac{dI}{dt} = (A - 1 - AI)I, \tag{1.12}$$

which exhibits a single quadratic nonlinearity.

There are other ways to non-dimensionalize the rate equations. Here time is measured in units of the photon damping time T_c but T_1 could equally be used to rescale time. It is also possible to introduce $2GT_cn$ and/or GT_1N as the dimensionless photon and population inversion variables. But the equations resulting from these normalizations are less appropriate for analysis than Eqs. (1.7) and (1.8). As previously emphasized, γ is small and it is mathematically convenient that it appears as a single parameter multiplying the right hand side of one of the two equations. Similar procedures have been applied for classical problems such as the van der Pol equation or the Michaelis–Menten equations in enzyme kinetics [8].

1.2 Steady states and linear stability

The analysis of our model equations starts with the determination of the steady states and their linear stability properties. The results allow us to predict bifurcations, anticipate interesting transient regimes, and possibly propose simplifications of the laser equations. The linear stability analysis is well documented for one- or two-variable systems of ordinary differential equations [26–28]. For higher order systems, we benefit from the Routh–Hurwitz conditions for the stability of the steady states ([26] p. 270, [29] p. 304).

1.2.1 Steady states

The steady state solutions of Eqs. (1.7) and (1.8) satisfy the conditions $dI/dt = dD/dt = 0$ or, equivalently, the following two equations for I and D

$$I(D - 1) = 0, \tag{1.13}$$

$$A - D(1 + I) = 0. \tag{1.14}$$

The possible solutions are (1) the zero intensity solution

$$I = 0 \quad \text{and} \quad D = A, \tag{1.15}$$

and (2) the non-zero intensity solution

$$I = A - 1 \geq 0 \quad \text{and} \quad D = 1. \tag{1.16}$$

The inequality in (1.16) is needed because I is an intensity. We conclude that the desired lasing action is possible only if $A > 1$. The critical point

$$(I, D, A) = (0, 1, 1) \tag{1.17}$$

is called the *laser first threshold* and is a *bifurcation point* because it connects our two steady state solutions. These solutions are represented as a function of the pump parameter A in Figure 1.2. The diagram is called a *bifurcation diagram* because it represents the amplitude of the possible solutions in terms of a control

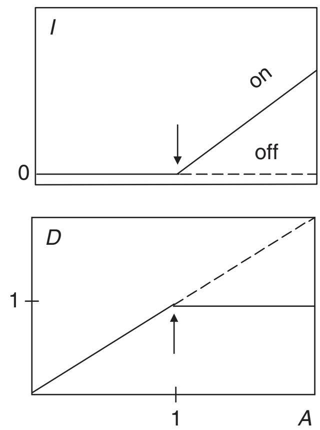

Fig. 1.2 Steady state solutions. Full and broken lines correspond to stable and unstable solutions, respectively. The arrow indicates the bifurcation point at $A = 1$.

or *bifurcation parameter*. In the zero intensity solution (laser OFF), the laser does not emit any radiation and the population difference sets to the value given by the pump ($D = A$). As the pump exceeds its threshold value $A = 1$, a non-zero intensity solution is possible (laser ON) and the laser emits radiation. The amount of emitted energy is proportional to the pump in excess of threshold, i.e. $I = A - 1$. Which of the two solutions will be effectively observed depends on their stability.

1.2.2 Linear stability

In order to analyze the stability of the steady states, we introduce the small deviations u and v defined by

$$u \equiv I - I_s \quad \text{and} \quad v \equiv D - D_s, \tag{1.18}$$

where $(I, D) = (I_s, D_s)$ denotes either OFF (1.15) or ON (1.16) solutions. We insert $I = I_s + u$ and $D = D_s + v$ into Eqs. (1.7) and (1.8), simplify by using the steady state equations (1.13) and (1.14), and neglect the quadratic terms in u and v. We then obtain the following *linearized equations* for u and v

$$\frac{du}{dt} = u(D_s - 1) + I_s v, \tag{1.19}$$

$$\frac{dv}{dt} = \gamma\,(-D_s u - (1 + I_s)v). \tag{1.20}$$

It is useful to rewrite these equations in matrix form as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.21}$$

where the 2×2 matrix J is called the Jacobian matrix and is defined here as

$$J \equiv \begin{pmatrix} D_s - 1 & I_s \\ -D_s\gamma & -(1 + I_s)\gamma \end{pmatrix}. \tag{1.22}$$

The general solution of Eqs. (1.19) and (1.20) or Eq. (1.21) is a linear combination of two exponential solutions. Introducing $u = c_1 \exp(\sigma t)$ and $v = c_2 \exp(\sigma t)$ into Eqs. (1.19) and (1.20) leads to a homogeneous system of two equations for c_1 and c_2. A nontrivial solution is possible only if the *growth rate* σ satisfies the *characteristic equation* given by

$$\det J - \sigma I = \sigma^2 + \sigma\left[\gamma(1 + I_s) - D_s + 1\right] + \gamma(1 + I_s - D_s) = 0. \tag{1.23}$$

Stability means that $\mathrm{Re}(\sigma_j) < 0$ $(j = 1, 2)$. Then the small deviations u and v will decay to zero. On the other hand, if $\mathrm{Re}(\sigma_j) > 0$ for either $j = 1$ or $j = 2$, u and v will grow exponentially and the steady state is unstable. The stability results are given as follows:

(1) For the zero intensity steady state (1.15), Eq. (1.23) admits the simple solutions

$$\sigma_1 = A - 1 \quad \text{and} \quad \sigma_2 = -\gamma. \tag{1.24}$$

From (1.24), we conclude that the zero intensity steady state is stable if $A < 1$ and unstable if $A > 1$.

(2) For the non-zero intensity steady state (1.16), Eq. (1.23) reduces to the following quadratic equation

$$\sigma^2 + \gamma A\sigma + \gamma(A - 1) = 0. \tag{1.25}$$

To determine the sign of $\mathrm{Re}(\sigma)$, we don't need to solve Eq. (1.25). Indeed, we note that the product of the roots is always positive $(\sigma_1\sigma_2 = \gamma(A - 1) > 0)$ and that the sum of the roots is always negative $(\sigma_1 + \sigma_2 = -\gamma A < 0)$. Together, the two inequalities imply that $\mathrm{Re}(\sigma_j) < 0$ $(j = 1, 2)$. Thus, the non-zero intensity solution is always stable.

At the bifurcation point (1.17), we note an exchange of stabilities between the zero intensity and non-zero intensity steady state solutions. This is a simple example of a *bifurcation with exchange of stability*. Some dynamical properties linked to the existence of this bifurcation will be examined in Section 1.5.

1.2.3 Damped relaxation oscillations

The linear stability analysis allows us to describe slowly decaying intensity oscillations that are observed in lasers after a sudden excitation such as a loss or gain pulse. Specifically, we solve the quadratic equation (1.25) and obtain

$$\sigma_{1,2} = -\gamma\frac{A}{2} \pm i\sqrt{\gamma(A - 1) - \gamma^2 A^2/4} \tag{1.26}$$

provided $\gamma(A - 1) - \gamma^2 A^2/4 \geq 0$. Expanding the two roots for small γ (A fixed) simplifies (1.26) as

$$\sigma_{1,2} = \pm i\sqrt{\gamma(A - 1)} - \gamma\frac{A}{2} + O(\gamma^{3/2}), \tag{1.27}$$

where the notation $O(\gamma^{3/2})$ means that the correction term is proportional to $\gamma^{3/2}$ (in Section 1.5.2, we examine the limit $A - 1$ small (γ fixed)). The meaning of the two first terms in (1.27) is best understood if we write the general solution for

$u = I - (A-1) = c\exp(\sigma_1 t) + \overline{c}\exp(\sigma_2 t)$, where $\overline{c}$ means the complex conjugate of c. Using (1.27), u can be rewritten as

$$u \simeq C\exp\left(-\gamma\frac{A}{2}t\right)\sin\left(\sqrt{\gamma(A-1)}t + \phi\right), \tag{1.28}$$

where C and ϕ are arbitrary constants determined by the initial conditions. The expression (1.28) implies that the intensity $I = A - 1 + u$ oscillates with a frequency proportional to $\sqrt{\gamma}$ and slowly decays with a rate proportional to γ. The frequency appearing in (1.28), defined by

$$\omega_R \equiv \sqrt{\gamma(A-1)}, \tag{1.29}$$

is called the laser *relaxation oscillation (RO) frequency* and is a reference frequency for all lasers experiencing intensity oscillations (see Problem 1.8.8 for the RO frequency close to threshold). The quantity

$$\Gamma \equiv \gamma\frac{A}{2} \tag{1.30}$$

is called the *damping rate* of the laser relaxation oscillations. Note that the expression (1.28) is the product of two functions that exhibit different time scales, namely[1]

$$t_1 = \sqrt{\gamma}t \quad \text{and} \quad t_2 = \gamma t. \tag{1.31}$$

In summary, the linearized theory reveals that the non-zero intensity steady state is weakly stable for all lasers exhibiting a small γ and that slowly decaying oscillations (RO oscillations) of the laser intensity are possible. Our results are strictly valid for small perturbations of the steady state. But in Section 5.2.1, we show that our conclusions remain valid if we consider arbitrary intensities.

1.3 Turn-on transients

In 1965, Pariser and Marshall [30] investigated the time evolution of the laser intensity using a He-Ne laser pumped by a flash lamp. The laser intensity was assumed to be initially close to zero and the time evolution was fitted using

$$\frac{dE}{dt} = aE - bE^3, \quad E(0) = E_0, \tag{1.32}$$

[1] In physical units, these two time scales are $\sqrt{T_1 T_c}T$ and $T_1 T$ respectively. The latter has a simple meaning since it coincides with the lifetime of the population inversion. The former is less obvious since it is the geometrical mean of two lifetimes. It appears because there exists a coupling between the variables I and D.

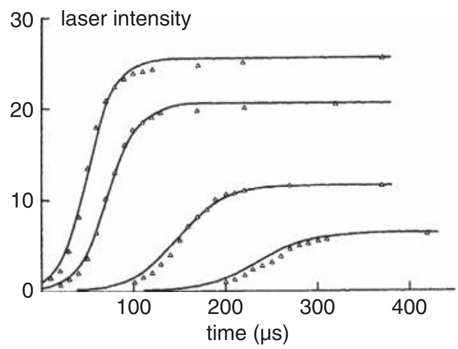

Fig. 1.3 He-Ne gas laser output as a function of time. From the lower to the upper time traces, the pump parameter above threshold is gradually increased. Reprinted Figure 2 with permission from Pariser and Marshall [30]. Copyright 1965 by the American Institute of Physics.

where E represents the electrical field and a and b are positive. This equation is the "class A" laser equation (1.12) with $I = E^2$, $a = (A - 1)/2$, and $b = A/2$. Equation (1.32) is a Bernoulli equation that can be solved exactly, leading to the following expression for the intensity I

$$I = \frac{a}{b} \frac{1}{1 - (1 - \frac{a}{bI_0}) \exp(-2at)}, \tag{1.33}$$

where $I_0 = E_0^2$. The different lines in Figure 1.3 correspond to (1.33) with different values of a and b. Note that a is proportional to the pump parameter above its threshold value. The expression (1.33) tells us that

$$\tau = (2a)^{-1} \tag{1.34}$$

is the time scale of the laser emission. It decreases as a increases (i.e. as the pump increases). Careful statistical studies of the laser build-up using a He-Ne laser [31] and a dye laser [32] complete the earlier investigations [30]. In both cases, Eq. (1.32) was used as the deterministic reference equation.

1.3.1 Typical turn-on experiment

For most common lasers used today in laboratories and in applications (solid state, CO_2, and semiconductor lasers), we switch the pump from a below- to an above-threshold value and observe the time evolution of the intensity. Figure 1.4 shows an example for a Nd^{3+}:YAG laser. We note three distinct regimes:

(1) A long time interval where the laser output power remains very low. In the conditions of Figure 1.4, this extends from the time origin given by the on-switching of pump