General Relativity: An Introduction for Physicists provides a clear mathematical introduction to Einstein's theory of general relativity. A wide range of applications of the theory are included, with a concentration on its physical consequences.

After reviewing the basic concepts, the authors present a clear and intuitive discussion of the mathematical background, including the necessary tools of tensor calculus and differential geometry. These tools are used to develop the topic of special relativity and to discuss electromagnetism in Minkowski spacetime. Gravitation as spacetime curvature is then introduced and the field equations of general relativity are derived. A wide range of applications to physical situations follows, and the conclusion gives a brief discussion of classical field theory and the derivation of general relativity from a variational principle.

Written for advanced undergraduate and graduate students, this approachable textbook contains over 300 exercises to illuminate and extend the discussion in the text.

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General Relativity

An Introduction for Physicists

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To our families

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Preface

General relativity is one of the cornerstones of classical physics, providing a synthesis of special relativity and gravitation, and is central to our understanding of many areas of astrophysics and cosmology. This book is intended to give an introduction to this important subject, suitable for a one-term course for advanced undergraduate or beginning graduate students in physics or in related disciplines such as astrophysics and applied mathematics. Some of the later chapters should also provide a useful reference for professionals in the fields of astrophysics and cosmology.

It is assumed that the reader has already been exposed to special relativity and Newtonian gravitation at a level typical of early-stage university physics courses. Nevertheless, a summary of special relativity from first principles is given in Chapter 1, and a brief discussion of Newtonian gravity is presented in Chapter 7. No previous experience of 4-vector methods is assumed. Some background in electromagnetism will prove useful, as will some experience of standard vector calculus methods in three-dimensional Euclidean space. The overall level of mathematical expertise assumed is that of a typical university mathematical methods course.

The book begins with a review of the basic concepts underlying special relativity in Chapter 1. The subject is introduced in a way that encourages from the outset a geometrical and transparently four-dimensional viewpoint, which lays the conceptual foundations for discussion of the more complicated spacetime geometries encountered later in general relativity. In Chapters 2–4 we then present a mini-course in basic differential geometry, beginning with the introduction of manifolds, coordinates and non-Euclidean geometry in Chapter 2. The topic of vector calculus on manifolds is developed in Chapter 3, and these ideas are extended to general tensors in Chapter 4. These necessary mathematical preliminaries are presented in such a way as to make them accessible to physics students with a background in standard vector calculus. A reasonable level of mathematical rigour has been maintained throughout, albeit accompanied by the occasional appeal to geometric intuition. The mathematical tools thus developed are then illustrated in Chapter 5 by re-examining the familiar topic of special relativity in a more formal manner, through the use of tensor calculus in Minkowski spacetime. These methods are further illustrated in Chapter 6, in which electromagnetism is described as a field theory in Minkowski spacetime, serving in some respects as a 'prototype' for the later discussion of gravitation. In Chapter 7, the incompatibility of special relativity and Newtonian gravitation is presented and the equivalence principle is introduced. This leads naturally to a discussion of spacetime curvature and the associated mathematics. The field equations of general relativity are then derived in Chapter 8, and a discussion of their general properties is presented.

The physical consequences of general relativity in a wide variety of astrophysical and cosmological applications are discussed in Chapters 9–18. In particular, the Schwarzschild geometry is derived in Chapter 9 and used to discuss the physics outside a massive spherical body. Classic experimental tests of general relativity based on the exterior Schwarzschild geometry are presented in Chapter 10. The interior Schwarzschild geometry and non-rotating black holes are discussed in Chapter 11, together with a brief mention of Kruskal coordinates and wormholes. In Chapter 12 we introduce two non-vacuum spherically symmetric geometries with a discussion of relativistic stars and charged black holes. Rotating objects are discussed in Chapter 13, including an extensive discussion of the Kerr solution. In Chapters 14–16 we describe the application of general relativity to cosmology and present a discussion of the Friedmann–Robertson–Walker geometry, cosmological models and the theory of inflation, including the generation of perturbations in the early universe. In Chapter 17 we describe linearised gravitation and weak gravitational fields, in particular drawing analogies with the theory of electromagnetism. The equations of linearised gravitation are then applied to the generation, propagation and detection of weak gravitational waves in Chapter 18. The book concludes in Chapter 19 with a brief discussion of classical field theory and the derivation of the field equations of electromagnetism and general relativity from variational principles.

Each chapter concludes with a number of exercises that are intended to illuminate and extend the discussion in the main text. It is strongly recommended that the reader attempt as many of these exercises as time permits, as they should give ample opportunity to test his or her understanding. Occasionally chapters have appendices containing material that is not central to the development presented in the main text, but may nevertheless be of interest to the reader. Some appendices provide historical context, some discuss current astronomical observations and some give detailed mathematical derivations that might otherwise interrupt the flow of the main text.

With regard to the presentation of the mathematics, it has to be accepted that equations containing partial and covariant derivatives could be written more compactly by using the comma and semi-colon notation, e.g. $v^a{}_{,b}$ for the partial derivative of a vector and $v^a{}_{;b}$ for its covariant derivative. This would certainly save typographical space, but many students find the labour of mentally unpacking such equations is sufficiently great that it is not possible to think of an equation's physical interpretation at the same time. Consequently, we have decided to write out such expressions in their more obvious but longer form, using $\partial_b v^a$ for partial derivatives and $\nabla_b v^a$ for covariant derivatives.

It is worth mentioning that this book is based, in large part, on lecture notes prepared separately by MPH and GPE for two different relativity courses in the Natural Science Tripos at the University of Cambridge. These courses were first presented in this form in the academic year 1999–2000 and are still ongoing. The course presented by MPH consisted of 16 lectures to fourth-year undergraduates in Part III Physics and Theoretical Physics and covered most of the material in Chapters 1–11 and 13–14, albeit somewhat rapidly on occasion. The course given by GPE consisted of 24 lectures to third-year undergraduates in Part II Astrophysics and covered parts of Chapters 1, 5–11, 14 and 18, with an emphasis on the less mathematical material. The process of combining the two sets of lecture notes into a homogeneous treatment of relativistic gravitation was aided somewhat by the fortuitous choice of a consistent sign convention in the two courses, and numerous sections have been rewritten in the hope that the reader will not encounter any jarring changes in presentational style. For many of the topics covered in the two courses mentioned above, the opportunity has been taken to include in this book a considerable amount of additional material beyond that presented in the lectures, especially in the discussion of black holes. Some of this material draws on lecture notes written by ANL for other courses in Part II and Part III Physics and Theoretical Physics. Some topics that were entirely absent from any of the above lecture courses have also been included in the book, such as relativistic stars, cosmology, inflation, linearised gravity and variational principles. While every care has been taken to describe these topics in a clear and illuminating fashion, the reader should bear in mind that these chapters have not been 'road-tested' to the same extent as the rest of the book.

It is with pleasure that we record here our gratitude to those authors from whose books we ourselves learnt general relativity and who have certainly influenced our own presentation of the subject. In particular, we acknowledge (in their current latest editions) S. Weinberg, *Gravitation and Cosmology*, Wiley, 1972; R. M. Wald, *General Relativity*, University of Chicago Press, 1984; B. Schutz, *A First Course in General Relativity*, Cambridge University Press, 1985; W. Rindler, *Relativity: Special, General and Cosmological*, Oxford University Press, 2001; and J. Foster & J. D. Nightingale, *A Short Course in General Relativity*, Springer-Verlag, 1995.

During the writing of this book we have received much help and encouragement from many of our colleagues at the University of Cambridge, especially members of the Cavendish Astrophysics Group and the Institute of Astronomy. In particular, we thank Chris Doran, Anthony Challinor, Steve Gull and Paul Alexander for numerous useful discussions on all aspects of relativity theory, and Dave Green for a great deal of advice concerning typesetting in LaTeX. We are also especially grateful to Richard Sword for creating many of the diagrams and figures used in the book and to Michael Bridges for producing the plots of recent measurements of the cosmic microwave background and matter power spectra. We also extend our thanks to the Cavendish and Institute of Astronomy teaching staff, whose examination questions have provided the basis for some of the exercises included. Finally, we thank several years of undergraduate students for their careful reading of sections of the manuscript, for pointing out misprints and for numerous useful comments. Of course, any errors and ambiguities remaining are entirely the responsibility of the authors, and we would be most grateful to have them brought to our attention. At Cambridge University Press, we are very grateful to our editor Vince Higgs for his help and patience and to our copy-editor Susan Parkinson for many useful suggestions that have undoubtedly improved the style of the book.

Finally, on a personal note, MPH thanks his wife, Becky, for patiently enduring many evenings and weekends spent listening to the sound of fingers tapping on a keyboard, and for her unending encouragement. He also thanks his mother, Pat, for her tireless support at every turn. MPH dedicates his contribution to this book to the memory of his father, Ron, and to his daughter, Tabitha, whose early arrival succeeded in delaying completion of the book by at least three months, but equally made him realise how little that mattered. GPE thanks his wife, Yvonne, for her support. ANL thanks all the students who have sat through his various lectures on gravitation and cosmology and provided useful feedback. He would also like to thank his family, and particularly his parents, for the encouragement and support they have offered at all times.