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170 Polynomials and Vanishing Cycles

POLYNOMIALS AND VANISHING CYCLES

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Frontmatter
More information

Contents

Preface

Vanishing cycles appear naturally in the picture when studying families of hypersurfaces, usually regarded as singular fibrations. The behaviour of vanishing cycles seems to be the cornerstone for understanding the geometry and topology of such families of spaces. There is a large literature, mostly over the last 40 years, showing the various ways in which vanishing cycles appear. For instance, we may associate to a holomorphic function f its sheaf of vanishing cycles, encoding information about the singularities and the monodromy of f.

Although quite sophisticated information is available (e.g. in Hodge theoretic terms, see the survey [Di2]), there are many open questions on the geometry of vanishing cycles (see for instance Donaldson's paper [Don] for an intriguing one).

This book proposes a systematic geometro-topological approach to the vanishing cycles appearing especially in *nonproper fibrations*. In such fibrations, some of the vanishing cycles do not correspond to the singularities on the space. Nevertheless, if the fibration extends to a proper one, then new singularities appear at the boundary and their relation to the original context may explain the presence of those vanishing cycles. The study of this type of problem in the setting of singular spaces and stratified singularities started notably with the works of Goresky and MacPherson, Hamm and Lê.

The situations where nonproper fibrations appear fall into two types are:
1. fibration on a noncompact space X, which is the restriction of a fibration over a given compact space Y such that $X = Y \setminus V$ for some subspace $V \subset Y$;
2. fibration on a noncompact X, which can be extended, nonuniquely, to a proper fibration on a larger space.

In case 1, the singularities of the given extension on the space Y are studied and then the information for the fibration on X are extracted. In case 2, first a 'good' candidate for the extension over some space Y should be found, and pursued as in case 1. For instance, let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be any polynomial function. This defines a nonproper fibration and can be extended to a meromorphic function $\tilde{f}/x_0^d : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^1$, where $\tilde{f}$ is the homogenization of f of degree $d = \deg f$, by the new variable x_0. Here, the embedding is $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$ and $\tilde{f}/x_0^d$ restricts to f on $\mathbb{C}^{n+1} = \mathbb{P}^{n+1} \setminus \{x_0 = 0\}$, but we may consider other embeddings.

The leading idea of this monograph is to bring into new light a bunch of topics – holomorphic germs, polynomial functions, pencils on quasi-projective spaces – conceiving them as aspects of a single theory with vanishing cycles at its core. A synthetic table with the topics and their relations is given in Figure 9. The new and highly general branches – meromorphic functions and non-generic Lefschetz pencils – complete and extend the landscape.

Parts I and II focus on complex polynomial functions f and discuss recent results in connection to the 'vanishing cycles at infinity' introduced in [ST2]. (Some aspects are discussed in real variables in Part I.) The specificity of the situation is the loss of 'information' toward infinity (e.g. singular points, curvature of fibres, vanishing cycles) and the aim is to explain the phenomena and to quantify this loss whenever possible. Roughly, the strategy is to compactify the family of fibres of f, study the proper extension of f especially at its singularities at infinity, and then derive the consequences of this study for the original affine setting.

Some evidence for the crucial importance of singularities at infinity in understanding the behaviour of polynomials is the famous unsolved Jacobian Conjecture. In two complex variables, an equivalent formulation of this conjecture is the following, cf. [LêWe, ST2]: *If $f : \mathbb{C}^2 \to \mathbb{C}$ has no critical points but has singularities at infinity, then, for any polynomial $h : \mathbb{C}^2 \to \mathbb{C}$, the zero locus $Z(\mathrm{Jac}(f, h))$ of the Jacobian is not empty*. Corollary 3.3.3 will show that, if the polynomial f has no critical points and no singularities at infinity, then all the fibres of f are CW-complexes with trivial homotopy groups, hence contractible. In this situation the Abhyankar–Moh–Suzuki theorem [AM, Suz] tells that f is linearizable, so the case left is indeed the one of singularities at infinity, as stated in the above conjecture.

Counting the vanishing cycles is an important issue and relates to enumerative geometry. In the complex setting, this is managed by an omnipresent character, the *polar curve*, to the role of which is dedicated Part II. Intersecting with the polar curve opens the way to counting points with multiplicities, which yields several invariants of the affine varieties, up to the embedding: CW-complex structure, relative homology groups, Euler obstruction, Chern–MacPherson cycle. Numerical polar invariants may control, under 'reasonable' circumstances, the behaviour of families of affine hypersurfaces or of polynomials: equisingularity at infinity, topological triviality, the integral of the curvature, the Gauss–Bonnet defect, etc. The geometry of polar curves enters into the study of the various aspects of the monodromy of f.

Part III studies the topology of pencils of hypersurfaces (or *meromorphic functions*) on stratified complex spaces. The context is a general one: *non-generic pencils*, which means pencils of hypersurfaces that may have singularities in the base locus (axis). This represents a unitary viewpoint on the Lefschetz–Morse–Zariski–Milnor theory, which is concerned with the change of topology when singularities occur while scanning the space by the levels of a function. Here, 'singularities' also means singularities at the boundary (whenever the space is open) and singularities in the axis of the pencil. This new standpoint, issued from [Ti8, Ti9, Ti12, Ti10], yields an extension of the classical context of *generic pencils*, also called *Lefschetz pencils*.

This book relies on the research I have done over the past 12 years, some of which was joint work. I owe very special thanks to Dirk Siersma. Several chapters stem from our joint papers [ST1-8] and handwritten notes, over which we have spent an immeasurable amount of time in Utrecht, in Lille, as RiP-ers in Oberwolfach and in many other places. I warmly thank my collaborators Alberto Verjovsky, Anatoly Libgober, José Seade, Alexandru Zaharia, Jörg Schürmann, Clément Caubel and Arnaud Bodin. Many results of our common papers were integrated into the book structure.

The monograph is intended for researchers and graduate students. The idea was to give transparent proofs, such that also nonspecialists can follow and get to grips with the literature. A list of exercises is provided at the end of almost every chapter, with a few hints at the appendix. I have privileged the self-containedness to the abundance of results. Besides the new presentation, there are also a few new results (Theorem 3.1.2, the determinacy scheme in Figure 1.1, Section 3.3, Proposition 4.1.5), improvements of some older statements, and a couple of new proofs in larger generality (e.g. Theorem 3.2.1, the global geometric monodromy in case of t-singularities §8.1).

As prerequisites, a good idea of differential and algebraic topology (homology, homotopy), and the basics on analytic and algebraic geometry are required. For singularity theory, some familiarity with Milnor's classical book [Mi2] is assumed. Reference is made to the appropriate literature whenever more involved results are needed for specialized topics. A list of some relevant textbooks and monographs is given at the end.

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