PART I

Singularities at infinity of polynomial functions

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Regularity conditions at infinity

1.1 Atypical values

Let $\mathbb{K}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $f:\mathbb{K}^n \to \mathbb{K}$ be a polynomial function of $n \geq 2$ variables. We denote by Sing$f$ the singular locus of f, that is the set of points $x \in \mathbb{K}^n$ such that the gradient $\mathrm{grad} f(x) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})(x)$ is equal to zero. Alternately, Sing$f$ is the set of points $x \in \mathbb{K}^n$ where f is not a submersion. It then follows by the implicit function theorem (submersion theorem) that f is differentially isotopic to a trivial fibration in some small neighbourhood of x. On a compact Riemannian manifold M (with or without boundary), the Ehresmann theorem tells us that we can get a fibration which is locally trivial on the target: if $v \in \mathbb{R}^p$ is a regular value of a function $h: M \to \mathbb{R}^p$, then this function is a trivial fibration over a small enough neighbourhood of v. In the situation of our polynomial function, this result cannot be applied. The reason is that we cannot control the trivialization in the 'neighbourhood of infinity'. This justifies the following definition.

Definition 1.1.1 We say that f *is topologically trivial at* $t_0 \in \mathbb{K}$ if there is a neighbourhood D of $t_0 \in \mathbb{K}$ such that the restriction $f_| : f^{-1}(D) \to D$ is a topologically trivial fibration. If t_0 does not satisfy this property, then we say that t_0 is an *atypical value* and that $f^{-1}(t_0)$ is an *atypical fibre*. We shall denote by Atyp$f$ the set of atypical fibres of f.

It is not so difficult to prove that the set $f(\mathrm{Sing} f)$ of critical values of f is a finite subset of $\mathbb{K}$. In the complex setting we have the inclusion (see Exercise 1.4):

$$f(\mathrm{Sing} f) \subset \mathrm{Atyp} f. \tag{1.1}$$

The inclusion can be strict even in very easy examples, such as the following one (given by Broughton [Br2]).

Example 1.1.2 $f:\mathbb{K}^2 \to \mathbb{K}$, $f(x,y) = x^2y + x$. We can quickly see that $\mathrm{Sing} f = \emptyset$. We have $f^{-1}(\varepsilon) = \{y = (\varepsilon - x)/x^2\}$ for $\varepsilon \neq 0$ and $f^{-1}(0) = \{x(xy+1) = 0\}$. Therefore, $f^{-1}(\varepsilon)$ is homeomorphic to $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$, whereas $f^{-1}(0)$ is homeomorphic to the disjoint union $\mathbb{K} \sqcup \mathbb{K}^*$. We obtain $\mathrm{Atyp} f \supset \{0\}$. This inclusion is actually an equality (Exercise 1.3).

It turns out that the set of atypical values of f is also finite. We shall give a proof in Corollary 1.2.13 after discussing several issues.[1] However the proofs of the finiteness of the set $\mathrm{Atyp} f$ are not constructive.

Among the natural problems that occur are the following ones:

to find a procedure to decide whether a noncritical value is atypical or not;
to describe how the topology of fibres changes at such a value.

In order to answer the first question, we would try to produce a trivialization at infinity as in Definition 1.1.1 by integrating the gradient vector field $\mathrm{grad} f$. It may then happen that some integral curves 'disappear' at infinity. This is due to the fact that $\mathrm{grad} f$ may tend to zero along nonbounded sequences of points. This phenomenon is well known in nonlinear analysis: we say that f does not satisfy the *Palais–Smale condition* [PaSm].

We shall explain two regularity conditions at infinity that go beyond the Palais–Smale condition: *ρ-regularity* and *t-regularity*. The former depends on the choice of a proper non negative C^1-function ρ, which defines a codimension one foliation in the neighbourhood of infinity. The latter condition depends on a compactification of f, but allows us to apply algebro-geometric tools, particularly efficient in the complex setting.

Triviality at infinity. Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial function of degree d. We consider the following algebraic subset of $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}$:

$$\mathbb{X}_{\mathbb{K}} := \{\tilde{f}(x_0, x) - t x_0^d = 0\} \subset \mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}, \tag{1.2}$$

where $\tilde{f}$ denotes the projectivization of f by the new variable x_0. Let:

$$\tau : \mathbb{X}_{\mathbb{K}} \to \mathbb{K}$$

be the projection to $\mathbb{K}$ and let us denote by $\mathbb{X}^{\infty}_{\mathbb{K}} := \mathbb{X}_{\mathbb{K}} \cap \{x_0 = 0\}$ the part at infinity of $\mathbb{X}_{\mathbb{K}}$.

Note 1.1.3 In case $\mathbb{K} = \mathbb{C}$ our set $\mathbb{X}_{\mathbb{C}}$ is precisely the closure in $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}$ of the graph $\{(x,t) \in \mathbb{K}^n \times \mathbb{K} \mid f(x) = t\}$ of f and the part at infinity $\mathbb{X}^{\infty}_{\mathbb{C}}$ is a divisor of $\mathbb{X}_{\mathbb{C}}$. It is clearly not so in the real case (example: $f(x,y) = x^4 + y^2$). This is one of the reasons why we shall stick to the complex setting later on.

We may identify $\mathbb{K}^n$ to $\mathbb{X} \setminus \mathbb{X}^\infty$ via the canonical map $x \overset{i}{\mapsto} ([x:1], f(x))$, which fits into the commuting diagram:

$$\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{i} & \mathbb{X} \\ f \searrow & & \swarrow \tau \\ & \mathbb{K} & \end{array} \tag{1.3}$$

In this way we get an extension of f which is a proper function.[2]

Definition 1.1.4 (**Local triviality at infinity**)
Let $y \in \mathbb{X}^\infty$ and let us denote by $B_\varepsilon \subset \mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}$ an open ball of radius ε centred at y and by $D_\delta \subset \mathbb{K}$ an open disk of radius δ centred at $\tau(y)$. (In the real setting D_δ means just a symmetric interval).

We say that f is *locally trivial at infinity, at* $y \in \mathbb{X}^\infty$, if there exists $\varepsilon_0 > 0$ for which the following condition holds: for any $0 < \varepsilon \leq \varepsilon_0$, there is a $\delta > 0$ such that the restriction:

$$\tau_| : (\mathbb{X} \setminus \mathbb{X}^\infty) \cap B_\varepsilon \cap \tau^{-1}(D_\delta) \to D_\delta \tag{1.4}$$

is a trivial topological fibration.

Definition 1.1.5 (**Topological triviality at infinity**)
We say that f is *topologically trivial at infinity at the value* $t_0 \in \mathbb{K}$ if there exists a compact set $K \subset \mathbb{K}^n$ and a disk D_δ centred at t_0 such that the restriction:

$$f_| : (\mathbb{K}^n \setminus K) \cap f^{-1}(D_\delta) \to D_\delta \tag{1.5}$$

is a trivial topological fibration.

If we replace D_δ by D^*_δ in the fibrations (1.4) and (1.5), then we can show that these are locally trivial fibrations, without any condition, see Appendix A1.1 and Theorem 3.1.6 respectively. A more precise notion, the *ϕ-controlled topological triviality*, will be given in Definition 3.1.8.

1.2 ρ-regularity and t-regularity

We introduce a regularity condition which is based on a control function. Let $K \subset \mathbb{K}^n$ be some compact (eventually empty) set and let:

$$\rho : \mathbb{K}^n \setminus K \to \mathbb{R}_{\geq 0}$$

be a proper C^1-submersion.

Definition 1.2.1 (**ρ-regularity at infinity**)
We say that f is *ρ-regular at* $y \in \mathbb{X}^\infty$ if there is an open ball $B_\varepsilon \subset \mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}$ centred at y and some disk $D_\delta \subset \mathbb{K}$ at $\tau(y)$ such that either $f^{-1}(D_\delta) \cap B_\varepsilon = \emptyset$ or, for all $c \in D_\delta$, the fibre $f^{-1}(c) \cap B_\varepsilon$ intersects all the levels of the restriction $\rho_{|B_\varepsilon \cap \mathbb{K}^n}$ and this intersection is transversal.

We say that the fibre $f^{-1}(t_0)$ is *ρ-regular at infinity* if f is ρ-regular at all points $y \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$. In this case we also say that t_0 is a ρ-regular value of f.

The transversality of the fibres of f to the levels of a control function recalls the well-known control functions ('fonction tapissante' or 'rug function', a notion due to Thom [Th1, Th2]) used by Thom and Mather in their First Isotopy Theorem along Whitney stratifications. More recently, the Isotopy Theorem has been proved for stratifications that are *(c)-regular*, a weaker regularity condition developed by Bekka [Be], which is also based on control functions.

If we use the Euclidean norm ρ_E in place of the function ρ in the above definition, then the ρ_E-regularity is a large-scale version[3] of the *transversality to small spheres*, the condition used by Milnor in the local study of holomorphic functions [Mi2, §4,5], see 3.1.

Remark 1.2.2 The fact that $f^{-1}(t_0)$ is ρ-regular at infinity is independent on the proper extension of f, since it is equivalent to the following: for any sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{K}^n$, $|x_k| \to \infty$, $f(x_k) \to t_0$, there exists some $k_0 = k_0((x_k)_{k\in\mathbb{N}})$ such that, if $k \geq k_0$, then f is transversal to ρ at x_k.

Example 1.2.3 $\rho : \mathbb{K}^n \to \mathbb{R}_{\geq 0}$, $\rho(x) = (\sum_{i=1}^n |x_i|^{2p_i})^{1/2p}$, where $(w_1, \ldots, w_n) \in \mathbb{N}^n$, $p = \operatorname{lcm}\{w_1, \ldots, w_n\}$ and $w_i p_i = p$, $\forall i$. This is a control function which can be used especially for polynomials f which are quasihomogeneous of type $(w_1, \ldots, w_n)$, see Exercise 1.7.

Proposition 1.2.4 If the fibre $f^{-1}(t_0)$ is ρ-regular at infinity, then f is topologically trivial at infinity at t_0.

Proof The set of points at infinity $\overline{f^{-1}(t_0)} \cap H^\infty$ is a compact set and therefore we can cover it by a finite number of balls B_i as in Definition 1.2.1. Let N be the union of these balls. Let D_i be the disk centred at t_0 which corresponds to the ball B_i in Definition 1.2.1, and let D be the smallest of those disks.

So N is a neighbourhood of $\tau^{-1}(t_0) \cap \mathbb{X}^\infty$ and we propose to show that the restriction $f_| : N \cap f^{-1}(D) \to D$ is a trivial fibration. Notice that Thom–Mather's Isotopy Theorem does not apply since the function $f_|$ is not proper. We use here ρ as global control function and construct a lift of the (real or complex) vector field $\partial/\partial t$ on D to a (real or complex) vector field **w** without zeros on $N \cap \mathbb{K}^n$, which is tangent to the levels $\rho =$ constant. We then get our topologically trivial fibration[4] by integrating **w**. For all the details of this type of construction we may refer the reader to Verdier's proof [Ve, Theorem 4.14] of Thom–Mather's Isotopy Theorem. □

Remark 1.2.5 In view of Definition 3.1.8, the proof of Proposition 1.2.4 yields the following more precise statement: *If the fibre $f^{-1}(t_0)$ is ρ-regular at infinity, then f is ρ-controlled topologically trivial at t_0.*

Corollary 1.2.6 If the fibre $f^{-1}(t_0)$ is nonsingular and ρ-regular at infinity, then $f^{-1}(t_0)$ is not an atypical fibre, i.e. $t_0 \notin \operatorname{Atyp} f$.

Proof From the proof of Proposition 1.2.4, we have a vector field **w** without zeros on a neighbourhood of infinity $N \cap \mathbb{K}^n$ and which is a lift of the unit vector field $\partial/\partial t$ on D.

Since the fibre $f^{-1}(t_0)$ is nonsingular, the gradient $\operatorname{grad} f$ has no zeros on $f^{-1}(t_0)$. Moreover, for any large ball $B \subset \mathbb{K}^n$ there exists a (small enough) disk D centred at t_0 such that $\operatorname{grad} f$ has no zeros on $B \cap f^{-1}(D)$. Therefore the vector field $\operatorname{grad} f$ is nowhere zero on $B \cap f^{-1}(D)$.

We then take a ball B such that $B \cap N \cap f^{-1}(D)$ is open and such that $(B \cup N) \cap f^{-1}(D) = f^{-1}(D)$. By a partition of unity, we glue the vector field **w** to the vector field $\mathbf{u} := \frac{\overline{\operatorname{grad} f}}{\| \operatorname{grad} f \|^2}$. The result is a vector field defined on $f^{-1}(D)$ which has the properties that it has no zeros and it is a lift of $\partial/\partial t$, since both vector fields **w** and **u** have these two properties. We then get a global trivialization over D by integrating this vector field. □

The relative conormal. Let $\mathcal{X} \subset \mathbb{K}^N$ be a $\mathbb{K}$-analytic variety. In the real case, assume that $\mathcal{X}$ contains at least a regular point. Let $U \subset \mathbb{K}^N$ be an open set and let $g : \mathcal{X} \cap U \to \mathbb{K}$ be $\mathbb{K}$-analytic and not constant. One calls *relative conormal of g* the subspace of the restriction of the cotangent bundle $T^*(\mathbb{K}^N)_{|\mathcal{X}\cap U}$ defined as follows[5]:

$$T^*_{g|\mathcal{X}\cap U} := \operatorname{closure}\{(y, \xi) \in T^*(\mathbb{K}^N) \mid y \in \mathcal{X}^0 \cap U, \xi(T_y(g^{-1}(g(y))) = 0\},$$

where $\mathcal{X}^0 \subset \mathcal{X}$ is the open dense subset of the regular points of $\mathcal{X}$ where g is a submersion. The relative conormal is *conical*, which means the following:

$$(y, \xi) \in T^*_{g|\mathcal{X}\cap U} \Rightarrow (y, \lambda\xi) \in T^*_{g|\mathcal{X}\cap U},\ \forall \lambda \in \mathbb{K}^*.$$

Let $\pi : T^*_{g|\mathcal{X}\cap U} \to \mathcal{X} \cap U$ denote the canonical projection and let $(T^*_{g|\mathcal{X}\cap U})_x := \pi^{-1}(x)$ for some $x \in \mathcal{X} \cap U$ such that $g(x) = 0$. We show that $(T^*_{g|\mathcal{X}\cap U})_x$

depends on the germ of g at x only up to multiplication by a unit in the analytic germ algebra $\mathcal{O}_{\mathcal{X},x}$.

Lemma 1.2.7 Let $\gamma : (\mathcal{X}, x) \to \mathbb{K}$ be $\mathbb{K}$-analytic and such that $\gamma(x) \neq 0$. Then $(T^*_{g|\mathcal{X}\cap U})_x = (T^*_{\gamma g|\mathcal{X}\cap U})_x$.

Proof Suppose first that $(\mathcal{X}, x)$ is nonsingular. We have $\operatorname{grad} \gamma g = \gamma \operatorname{grad} g + g \operatorname{grad} \gamma$, hence:

$$\frac{\operatorname{grad} \gamma g}{\| \operatorname{grad} g \|} = \gamma \frac{\operatorname{grad} g}{\| \operatorname{grad} g \|} + \operatorname{grad} \gamma \frac{g}{\| \operatorname{grad} g \|}.$$

Since γ is analytic, $\| \operatorname{grad} \gamma \|$ and γ are bounded within some neighbourhood of x. We have the following inequality due to Łojasiewicz [Łoj1]:

$$\| \operatorname{grad} g \| \geq |g|^{\theta}, \quad \text{for some } 1 > \theta > 0,$$

which is valid in some small enough neighbourhood of x. Since $g(x) = 0$ we get that $\frac{g(y)}{\| \operatorname{grad} g(y) \|}$ tends to zero as the point y tends to x. Therefore, along any sequence of points tending to x, we have $\lim \frac{\operatorname{grad} \gamma g}{\| \operatorname{grad} g \|} = \gamma(x) \lim \frac{\operatorname{grad} g}{\| \operatorname{grad} g \|}$. This shows that the limits of the directions $\operatorname{grad} \gamma g$ and $\operatorname{grad} g$ are the same.

Let $g^{-1}(0)$ be denoted by $\mathcal{Y}$. In the general case we resolve $\mathcal{X}$ within an embedded resolution $p : \tilde{\mathcal{X}} \to \mathcal{X}$, to a smooth variety $\tilde{\mathcal{X}}$. This is an isomorphism over $\mathcal{X}_{\text{reg}} \setminus \mathcal{Y}$. Now apply the result proved above in the nonsingular case to the functions $g \circ p$ and $(\gamma \circ p)(g \circ p)$, then pull down to the conormal of $\mathcal{X}$, by the following:

$$\begin{array}{ccccc} (T^*\mathbb{K}^N)_{|\tilde{\mathcal{X}}\cap p^{-1}(U)} & \hookleftarrow & p^*(T^*\mathbb{K}^N)_{|\mathcal{X}\cap U} & \to & (T^*\mathbb{K}^N)_{|\mathcal{X}\cap U} \\ & \searrow & \downarrow & & \downarrow \pi \\ & & \tilde{\mathcal{X}} \cap p^{-1}(U) & \stackrel{p}{\to} & \mathcal{X} \cap U. \end{array}$$

□

Remark 1.2.8 In the above proof, it is not necessary that γ is an analytic function, it is only important that γ and $\| \operatorname{grad} \gamma \|$ are bounded.

Let us come back to our polynomial function f and its associated space $\mathbb{X}_{\mathbb{K}} \subset \mathbb{P}^n \times \mathbb{K}$. The subspace $\mathbb{X}^{\infty}_{\mathbb{K}} \subset \mathbb{X}_{\mathbb{K}}$ is covered by the union of the affine charts $U_i \times \mathbb{K}$, where $U_i := \{x_i \neq 0\}$, for $0 < i \leq n$. In each chart, the subspace $\mathbb{X}^{\infty}_{\mathbb{K}}$ is defined by the equation $x_0 = 0$ and therefore the relative conormal $T^*_{x_0|\mathbb{X}\cap(U_i\times\mathbb{K})}$ is well defined. The equations $x_0 = 0$ differ from one chart to the other by multiplication with a rational function of type x_i/x_j.

Since this function is nonzero on $\mathbb{X}^\infty_\mathbb{K}$, we deduce from Lemma 1.2.7 that the fibre $(T^*_{x_0|\mathbb{X}\cap(U_i\times\mathbb{K})})_y$ is independent on the chart and therefore we may write $(T^*_{x_0|\mathbb{X}})_y$. We then have the following definition:

Definition 1.2.9 We say that (y,ξ) is a *characteristic covector at infinity* if $\xi \in (T^*_{x_0|(\mathbb{X}\setminus\mathbb{X}^\infty)\cap U})_y$, where $y \in \mathbb{X}^\infty_\mathbb{K} \cap \overline{f^{-1}(\tau(y))}$ and $U \subset \mathbb{X}$ is some neighbourhood of y. We denote by $\mathcal{C}^\infty_\mathbb{K}$ the *subspace of characteristic covectors at infinity*.

We have proved above that $\mathcal{C}^\infty_\mathbb{K}$ is an analytic subspace of the restriction of the cotangent bundle $T^*(\mathbb{P}^n \times \mathbb{K})$ over $\mathbb{X}^\infty_\mathbb{K}$. We also note that $\mathcal{C}^\infty_\mathbb{K}$ is conical* and therefore we may consider its projectivization $\mathbb{P}(\mathcal{C}^\infty_\mathbb{K})$. With these definitions, let us introduce the announced regularity condition.

Definition 1.2.10 (***t*-regularity at infinity; *t*-singularities,** $\mathrm{Sing}^\infty f$)
The fibre $f^{-1}(t_0)$ (or that f) is *t-regular* at $y \in \mathbb{X}^\infty_\mathbb{K} \cap \overline{f^{-1}(t_0)}$ iff $(y, \mathrm{d}t)$ is not a characteristic covector at infinity, i.e. $(y, \mathrm{d}t) \notin \mathcal{C}^\infty_\mathbb{K}$. We also say that $f^{-1}(t_0)$ is *t-regular at infinity* if this fibre is t-regular at all its points at infinity.

We call *t-singularities of f at infinity* the points $y \in \mathbb{X}^\infty_\mathbb{K}$ where f is not t-regular. We denote by $\mathrm{Sing}^\infty f$ the set of t-singularities of f at infinity.

Remark 1.2.11 It follows from the definition that, if $y \in \overline{\mathrm{Sing} f} \cap \mathbb{X}^\infty_\mathbb{K}$, then f is not t-regular at y. The reciprocal is obviously not true (Example 1.1.2) and therefore we need to investigate more closely what is the set $\mathrm{Sing}^\infty f$.

We now relate the t-regularity to the ρ-regularity. Let us denote by ρ_E the Euclidean norm in $\mathbb{K}^n$. This is a control function and the ρ_E-regularity is well defined.

Proposition 1.2.12 If f is t-regular at $y \in \mathbb{X}^\infty_\mathbb{K}$, then f is ρ_E-regular at y. In particular, if $f^{-1}(t_0)$ is t-regular at infinity, then $f^{-1}(t_0)$ is ρ_E-regular at infinity.

Proof Let $d^\infty : \mathbb{X}_\mathbb{K} \to \mathbb{R}$ be defined by:

$$\begin{cases} d^\infty(x, f(x)) = 1/\rho_E^2(x), & \text{for } x \in \mathbb{K}^n \\ d^\infty(y) = 0, & \text{for } y \in \mathbb{X}^\infty_\mathbb{K}. \end{cases}$$

By computing in local charts, we can see that d^∞ is a rational function. (After Thom [Th1], this is an example of a *rug function*.) Moreover, in the neighbourhood of some point $y \in \mathbb{X}^\infty_\mathbb{K}$, the functions $|x_0|^2$ and d^∞ differ by a nonzero factor and have the same zero locus, the germ of $\mathbb{X}^\infty_\mathbb{K}$ at y. By Lemma 1.2.7 we have:

$$(T^*_{d^\infty|\mathbb{X}})_y = (T^*_{|x_0|^2|\mathbb{X}})_y. \tag{1.6}$$

* In the complex setting, $\mathcal{C}^\infty_\mathbb{C}$ is also a *Lagrangean* subvariety of $T^*(\mathbb{P}^n \times \mathbb{C})_{|\mathbb{X}^\infty}$.

Let us consider the real setting first. Then $(T^*_{|x_0|^2|\mathbb{X}})_y = (T^*_{x_0|\mathbb{X}})_y$ and the latter is in turn equal to $(\mathcal{C}^\infty_{\mathbb{R}})_y$. The condition $(y, \mathrm{d}t) \notin (T^*_{d^\infty|\mathbb{X}})_y$ is therefore equivalent to the t-regularity at $y \in \mathbb{X}^\infty$. On the other hand, it implies that, in some neighbourhood of y intersected with $\mathbb{K}^n$, the fibres $t =$ constant are transversal to the levels of the function d^∞. These levels are the same as the levels of the function ρ_E, so this finishes the proof in the real case.

The complex case $\mathbb{K} = \mathbb{C}$ now. By the conical structure of $T^*(\mathbb{K}^n)$ we may take the projectivization $\mathbb{P}T^*(\mathbb{K}^n)$, which is the quotient space by the action of $\mathbb{K}^*$. Let us introduce the map $\iota : \mathbb{P}T^*(\mathbb{R}^{2n}) \to \mathbb{P}T^*(\mathbb{C}^n)$ between the real and the complex projectivized conormal bundles (where $\mathbb{R}^{2n}$ is the real underlying space of $\mathbb{C}^n$) defined as follows: if ξ is conormal to a hyperplane $H \subset \mathbb{R}^{2n}$, then $\iota([\xi])$ is the conormal to the unique complex hyperplane included in H. This is clearly a continuous map. We then have the following equality:

$$\mathbb{P}(\mathcal{C}^\infty_{\mathbb{C}})_y = \iota(\mathbb{P}(T^*_{|x_0|^2|\mathbb{X}})_y),$$

since the complex tangent space $T_x\{x_0 = \text{constant}\}$ is exactly the unique complex hyperplane contained in the real tangent space $T_x\{|x_0|^2 = \text{constant}\}$. The equality follows by the fact that ι commutes with taking limits. Now $(y, \mathrm{d}t) \notin (\mathcal{C}^\infty_{\mathbb{C}})_y$ implies $(y, \iota^{-1}([\mathrm{d}t]) \notin \mathbb{P}(T^*_{|x_0|^2|\mathbb{X}})_y$. This implies the ρ_E-regularity at y since the equality (1.6) is true in the complex setting too. □

We may now give a proof of the finiteness of the set of nonregular values, based on Whitney stratifications of semi-algebraic sets (see §A1.1).

Corollary 1.2.13 Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial. The set $\tau(\mathrm{Sing}^\infty f)$ of the values $t_0 \in \mathbb{K}$ such that $f^{-1}(t_0)$ is not t-regular at infinity is a finite set.

In particular, the set of values of f that are not ρ_E-regular at infinity is a finite set.

Proof There exists a Whitney stratification $\mathcal{W} = \{\mathcal{W}_i\}_i$ of $\mathbb{X}_{\mathbb{K}}$ with $\mathbb{K}^n$ as a stratum and with a finite number of strata, which has in addition the Thom property with respect to the function x_0 in any local chart.* We call it a *Thom–Whitney stratification at infinity*.[6]

If $\tau^{-1}(t_0)$ is transversal to a stratum $\mathcal{W}_i \subset \mathbb{X}^\infty$ at some point y, then $\tau^{-1}(t_0)$ is transversal to the limits of the tangents spaces to the levels of x_0, by the assumed (a_{x_0})-property of our Thom–Whitney stratification. Therefore $f^{-1}(t_0)$ is transversal to the levels of x_0 in the neighbourhood of y and hence f is t-regular at infinity at y (see also §2.2 and (2.6)).

* See §A1.1 and also the proof of Proposition 2.2.3.

The restriction of the projection $\tau : \mathbb{X}_{\mathbb{K}} \to \mathbb{K}$ to a stratum contained in $\mathbb{X}^{\infty}$ has a finite number of critical values: since strata are semi-algebraic, the set of critical values of τ is semi-algebraic, discrete, and has a finite number of connected components.* Since the number of strata is itself finite, this implies that the set of values t_0 such that $\tau^{-1}(t_0)$ is not transversal to all the strata it meets, is a finite set. By the above discussion, this contains the set $\tau(\mathrm{Sing}^{\infty} f)$.

The second claim follows from Proposition 1.2.12. □

Corollary 1.2.14 The set $\mathrm{Atyp} f$ of atypical values of f is a finite set.

Proof We have the inclusion:

$$\mathrm{Atyp} f \subset \tau(\mathrm{Sing} f \cup \mathrm{Sing}^{\infty} f). \tag{1.7}$$

Indeed, this follows from Corollary 1.2.6 by the fact that t-regularity implies ρ_E-regularity (Proposition 1.2.12). Now, the set $\tau(\mathrm{Sing}^{\infty} f)$ is finite (Corollary 1.2.13) and the set $\mathrm{Sing} f$ is finite too (Exercise 1.2). □

1.3 The Malgrange condition

We introduce another regularity condition, which is more computable and which will turn out to be equivalent to the t-regularity.[7]

Definition 1.3.1 Let $\{p_j\}_{j\in\mathbb{N}}$ be a sequence of points in $\mathbb{K}^n$ and let us consider the following properties:

(L_1) $\|p_j\| \to \infty$ and $f(p_j) \to t_0$, as $j \to \infty$.
(L_2) $p_j \to y \in \mathbb{X}^{\infty}_{\mathbb{K}}$, as $j \to \infty$.

We say that the fibre $f^{-1}(t_0)$ verifies the *Malgrange condition* if there is $\delta > 0$ such that, for any sequence of points with property (L_1), we have:

$$\|p_j\| \cdot \|\operatorname{grad} f(p_j)\| > \delta. \tag{1.8}$$

We say that f verifies the Malgrange condition at $y \in \mathbb{X}^{\infty}_{\mathbb{K}}$ if there is $\delta > 0$ such that (1.8) holds for any sequence of points with property (L_2).

Right from the definition, it follows that $f^{-1}(t_0)$ verifies the Malgrange condition if and only if f verifies the Malgrange condition at any point $y \in \tau^{-1}(t_0) \cap \mathbb{X}^{\infty}_{\mathbb{K}}$. We have the following characterization.

* This uses Tarki–Seidenberg theorem and Whitney's finiteness theorem, see for instance [GWPL].