Part One

NEWTONIAN MECHANICS OF A SINGLE PARTICLE

CHAPTERS IN PART ONE

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Excerpt
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Chapter One

The algebra and calculus of vectors

KEY FEATURES

The key features of this chapter are the **rules of vector algebra** and **differentiation of vector functions** of a scalar variable.

This chapter begins with a review of the rules and applications of **vector algebra**. Almost every student taking a mechanics course will already have attended a course on vector algebra, and so, instead of covering the subject in full detail, we present, for easy reference, a summary of vector operations and their important properties, together with a selection of worked examples.

The chapter closes with an account of the **differentiation of vector functions** of a scalar variable. Unlike the vector algebra sections, this is treated in full detail. Applications include the **tangent vector** and **normal vector** to a curve. These will be needed in the next chapter in order to interpret the velocity and acceleration vectors.

1.1 VECTORS AND VECTOR QUANTITIES

Most physical quantities can be classified as being **scalar quantities** or **vector quantities**. The temperature in a room is an example of a scalar quantity. It is so called because its *value* is a scalar, which, in the present context, means a *real number*. Other examples of scalar quantities are the volume of a can, the density of iron, and the pressure of air in a tyre. Vector quantities are defined as follows:

Definition 1.1 ***Vector quantity*** *If a quantity Q has a* ***magnitude*** *and a* ***direction*** *associated with it, then Q is said to be a* ***vector quantity***. [Here, magnitude means a positive real number and direction is specified relative to some underlying reference frame* that we regard as fixed.]

The **displacement** of a particle† is an example of a vector quantity. Suppose the particle starts from the point A and, after moving in a general manner, ends up at the

* See section 2.2 for an explanation of the term 'reference frame'.

† A particle is an idealised body that occupies only a single point of space.

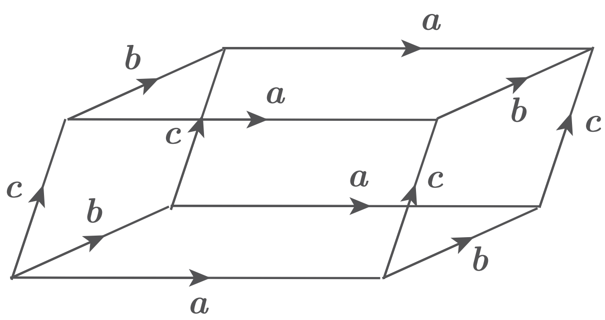

FIGURE 1.1 Four different representations of each of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ $\boldsymbol{c}$ form the twelve edges of the parallelepiped box.

point B. The *magnitude* of the displacement is the distance AB and the *direction* of the displacement is the direction of the straight line joining A to B (in that order). Another example is the **force** applied to a body by a rope. In this case, the *magnitude* is the strength of the force (a real positive quantity) and the *direction* is the direction of the rope (away from the body). Other examples of vector quantities are the velocity of a body and the value of the electric (or magnetic) field. In order to manipulate all such quantities without regard to their physical origin, we introduce the concept of a vector as an *abstract quantity*.

Definition 1.2 ***Vector*** *A* ***vector*** *is an* ***abstract*** *quantity characterised by the two properties* ***magnitude*** *and* ***direction****. Thus two vectors are equal if they have the same magnitude and the same direction.**

Notation. Vectors are written in bold type, for example $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{r}$ or $\boldsymbol{F}$. The **magnitude** of the vector $\boldsymbol{a}$, which is a real positive number, is written $|\boldsymbol{a}|$, or sometimes† simply a.

It is convenient to define operations involving abstract vectors by reference to some simple, easily visualised vector quantity. The standard choice is the set of directed **line segments**. Each straight line joining two points (P and Q say, in that order) is a vector quantity, where the magnitude is the distance PQ and the direction is the direction of Q relative to P. We call this the line segment $\overrightarrow{PQ}$ and we say that it *represents* some abstract vector $\boldsymbol{a}$.‡ Note that each vector $\boldsymbol{a}$ is represented by infinitely many different line segments, as indicated in Figure 1.1.

* In order that our set of vectors should have a standard algebra, we also include a special vector whose magnitude is zero and whose direction is not defined. This is called the **zero vector** and written **0**. The zero vector is not the same thing as the number zero!

† It is often useful to denote the magnitudes of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, ... by a, b, c, ..., but this does risk confusion. Take care!

‡ The zero vector is represented by line segments whose end point and starting point are coincident.

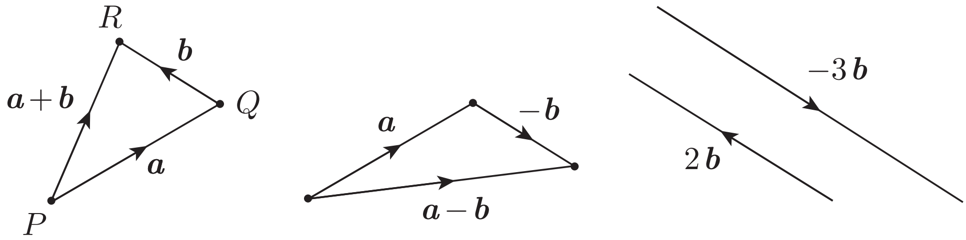

FIGURE 1.2 Addition, subtraction and scalar multiplication of vectors.

1.2 LINEAR OPERATIONS: $\boldsymbol{a}+\boldsymbol{b}$ AND $\lambda\boldsymbol{a}$

Since vectors are abstract quantities, we can define sums and products of vectors in any way we like. However, in order to be of any use, the definitions must create some coherent algebra and represent something of interest when applied to a range of vector quantities. Also, our definitions must be independent of the particular representations used to construct them. The definitions that follow satisfy all these requirements.

The vector sum $\boldsymbol{a}+\boldsymbol{b}$

Definition 1.3 ***Sum of vectors*** *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be any two vectors. Take any representation* $\overrightarrow{PQ}$ *of* $\boldsymbol{a}$ *and suppose the line segment* $\overrightarrow{QR}$ *represents* $\boldsymbol{b}$. *Then the* ***sum*** $\boldsymbol{a}+\boldsymbol{b}$ *of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is the* ***vector*** *represented by the line segment* $\overrightarrow{PR}$, *as shown in Figure 1.2 (left).*

Laws of algebra for the vector sum

(i) $\boldsymbol{b}+\boldsymbol{a}=\boldsymbol{a}+\boldsymbol{b}$ (commutative law)

(ii) $\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$ (associative law)

Definition 1.4 ***Negative of a vector*** *Let* $\boldsymbol{b}$ *be any vector. Then the vector with the same magnitude as* $\boldsymbol{b}$ *and the* ***opposite*** *direction is called the* ***negative*** *of* $\boldsymbol{b}$ *and is written* $-\boldsymbol{b}$. ***Subtraction*** *by* $\boldsymbol{b}$ *is then defined by*

$$\boldsymbol{a}-\boldsymbol{b}=\boldsymbol{a}+(-\boldsymbol{b}).$$

[That is, to subtract $\boldsymbol{b}$ just add $-\boldsymbol{b}$, as shown in Figure 1.2 (centre).]

The scalar multiple $\lambda\boldsymbol{a}$

Definition 1.5 ***Scalar multiple*** *Let* $\boldsymbol{a}$ *be a vector and* λ *be a scalar (a real number). Then the* ***scalar multiple*** $\lambda\boldsymbol{a}$ *is the vector whose magnitude is* $|\lambda||\boldsymbol{a}|$ *and whose direction is*

(i) *the same as* $\boldsymbol{a}$ *if* λ *is positive,*
(ii) *undefined if* λ *is zero (the answer is the zero vector),*
(iii) *the same as* $-\boldsymbol{a}$ *if* λ *is negative.*

It follows that $-(\lambda\boldsymbol{a}) = (-\lambda)\boldsymbol{a}$.

Laws of algebra for the scalar multiple

(i) $\lambda(\mu\boldsymbol{a}) = (\lambda\mu)\boldsymbol{a}$ (associative law)

(ii) $\lambda(\boldsymbol{a}+\boldsymbol{b}) = \lambda\boldsymbol{a}+\lambda\boldsymbol{b}$ and $(\lambda+\mu)\boldsymbol{a} = \lambda\boldsymbol{a}+\mu\boldsymbol{a}$ (distributive laws)

The effect of the above laws is that **linear combinations** of vectors can be manipulated just *as if* the vectors were symbols representing real or complex numbers.

Example 1.1 *Laws for vector sum and scalar multiple*

Simplify the expression $3(2\boldsymbol{a}-4\boldsymbol{b}) - 2(2\boldsymbol{a}-\boldsymbol{b})$.

Solution

On this one occasion we will do the simplification by strict application of the laws. It is instructive to decide which laws are being used at each step!

$$\begin{aligned}3(2\boldsymbol{a}-4\boldsymbol{b}) - 2(2\boldsymbol{a}-\boldsymbol{b}) &= 3\Big(2\boldsymbol{a}+(-4)\boldsymbol{b}\Big) + (-2)\Big(2\boldsymbol{a}+(-1)\boldsymbol{b}\Big)\\ &= \Big(6\boldsymbol{a}+(-12)\boldsymbol{b}\Big) + \Big((-4)\boldsymbol{a}+2\boldsymbol{b}\Big)\\ &= \Big(6\boldsymbol{a}+(-4)\boldsymbol{a}\Big) + \Big((-12)\boldsymbol{b}+2\boldsymbol{b}\Big)\\ &= 2\boldsymbol{a}+(-10)\boldsymbol{b} = 2\boldsymbol{a}-10\boldsymbol{b}. \blacksquare\end{aligned}$$

Unit vectors

A vector of **unit magnitude** is called a **unit vector**. If any vector $\boldsymbol{a}$ is divided by its own magnitude, the result is a *unit vector* having the *same direction* as $\boldsymbol{a}$. This new vector is denoted by $\widehat{\boldsymbol{a}}$ so that

$$\widehat{\boldsymbol{a}} = \boldsymbol{a}/|\boldsymbol{a}|.$$

Basis sets

Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are two non-zero vectors, with the direction of $\boldsymbol{b}$ neither the same nor opposite to that of $\boldsymbol{a}$. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ be representations of $\boldsymbol{a}$, $\boldsymbol{b}$ and let $\mathcal{P}$ be the plane

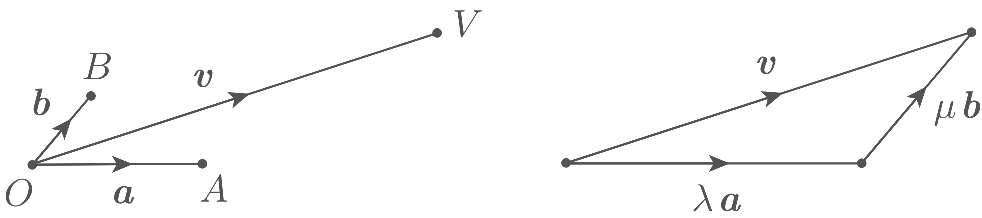

FIGURE 1.3 The set $\{\boldsymbol{a}, \boldsymbol{b}\}$ is a basis for all vectors lying in the plane OAB.

containing the triangle OAB. Then (see Figure 1.3) any vector $\boldsymbol{v}$ whose representation $\overrightarrow{OV}$ lies in the plane $\mathcal{P}$ can be written in the form

$$\boldsymbol{v} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}, \tag{1.1}$$

where the coefficients λ, μ are unique. Vectors that have their directions parallel to the same plane are said to be **coplanar**. Thus we have shown that *any vector coplanar with* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *can be expanded uniquely in the form* (1.1). It is also apparent that this expansion set cannot be reduced in number (in this case to a single vector). For these reasons the pair of vectors $\{\boldsymbol{a}, \boldsymbol{b}\}$ is said to be a **basis set** for vectors lying* in the plane $\mathcal{P}$.

Suppose now that $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a set of three non-coplanar vectors. Then any vector $\boldsymbol{v}$, *without restriction*, can be written in the form

$$\boldsymbol{v} = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c}, \tag{1.2}$$

where the coefficients λ, μ, ν are unique. In this case we say that the set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a **basis set** for all three-dimensional vectors. Although *any* set of three non-coplanar vectors forms a basis, it is most convenient to take the basis vectors to be *orthogonal unit vectors*. In this case the basis set† is usually denoted by $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ and is said to be an **orthonormal basis**. The representation of a general vector $\boldsymbol{v}$ in the form

$$\boldsymbol{v} = \lambda\boldsymbol{i} + \mu\boldsymbol{j} + \nu\boldsymbol{k}$$

is common in problem solving.

In applications involving the cross product of vectors, the distinction between **right-** and **left-handed** basis sets actually matters. There is no experiment in classical mechanics or electromagnetism that can distinguish between right- and left-handed sets. The difference can only be exhibited by a model or some familiar object that exhibits 'handedness', such as a corkscrew.‡ Figure 1.4 shows a **right-handed orthonormal basis set** attached to a well known object.

* Strictly speaking vectors are abstract quantities that do not *lie* anywhere. This phrase should be taken to mean '*vectors whose directions are parallel to the plane* $\mathcal{P}$'.

† It should be remembered that there are *infinitely* many basis sets made up of orthogonal unit vectors.

‡ Suppose that the non-coplanar vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ have representations $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ respectively. Place an ordinary corkscrew with the screw lying along the line through O perpendicular to the plane OAB,

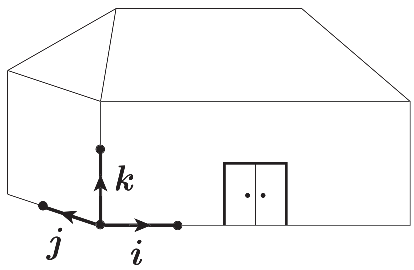

FIGURE 1.4 A **standard basis** set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is both **orthonormal** and **right-handed**.

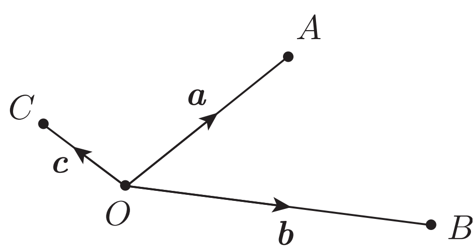

FIGURE 1.5 The points A, B, C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ relative to the origin O.

Definition 1.6 ***Standard basis set*** *If an* ***orthonormal basis*** $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ *is also* ***right-handed*** *(as shown in Figure 1.4), we will call it a* ***standard basis***.

Position vectors and vector geometry

Suppose that O is a fixed point of space. Then relative to the origin O (and relative to the underlying reference frame), any point of space, such as A, has an associated line segment, $\overrightarrow{OA}$, which represents some vector $\boldsymbol{a}$. Conversely, the vector $\boldsymbol{a}$ is sufficient to specify the position of the point A.

Definition 1.7 ***Position vector*** *The vector* $\boldsymbol{a}$ *is called the* ***position vector*** *of the point A relative to the origin O, [It is standard practice, and very convenient, to denote the position vectors of the points A, B, C, ... by* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, *and so on, as shown in Figure 1.5.]*

Since vectors can be used to specify the positions of points in space, we can now use the laws of vector algebra to prove* results in Euclidean geometry. This is not just an academic exercise. Familiarity with geometrical concepts is an important part of mechanics. We begin with the following useful result:

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and the handle parallel to OA. Now turn the corkscrew until the handle is parallel to OB and note the direction in which the corkscrew *would* move if it were 'in action'. (The direction of the turn must be such that the angle turned through is at most 180°.) If OC makes an *acute angle* with this direction, the set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ (in that order) is *right-handed*; if OC makes an *obtuse* angle with this direction then the set is *left-handed*.

* Some properties of Euclidean geometry have been used to prove the laws of vector algebra. However, this does not prevent us from giving valid proofs of other results.

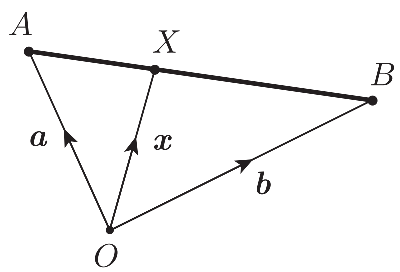

FIGURE 1.6 The point X divides the line AB in the ratio $\lambda : \mu$.

Example 1.2 *Point dividing a line in a given ratio*

The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ relative to an origin O. Find the position vector $\boldsymbol{x}$ of the point X that divides the line AB in the ratio $\lambda : \mu$ (that is $AX/XB = \lambda/\mu$).

Solution

It follows from Figure 1.6 that $\boldsymbol{x}$ is given by*

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{a} + \overrightarrow{AX} = \boldsymbol{a} + \left(\frac{\lambda}{\lambda+\mu}\right)\overrightarrow{AB}\\ &= \boldsymbol{a} + \left(\frac{\lambda}{\lambda+\mu}\right)(\boldsymbol{b}-\boldsymbol{a}) = \frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}.\end{aligned}$$

In particular, the **mid-point** of the line AB has position vector $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$. ■

Example 1.3 *Centroid of a triangle*

Show that the three medians of any triangle meet in a point (the centroid) which divides each of them in the ratio 2:1.

Solution

Let the triangle be ABC where the points A, B, C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ relative to some origin O. Then the mid-point P of the side BC has position vector $\boldsymbol{p} = \frac{1}{2}(\boldsymbol{b}+\boldsymbol{c})$. The point X that divides the median AP in the ratio 2:1 therefore has position vector

$$\boldsymbol{x} = \frac{\boldsymbol{a}+2\boldsymbol{p}}{2+1} = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{3}.$$

The position vectors of the corresponding points on the other two medians can be found by cyclic permutation of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and clearly give the same value. Hence all three points are coincident and so the three medians meet there. ■

* Strictly speaking we should not write expressions like $\boldsymbol{a}+\overrightarrow{AX}$ since the sum we defined was the sum of two *vectors*, not a vector and a line segment. What we really mean is '*the sum of* $\boldsymbol{a}$ *and the vector represented by the line segment* $\overrightarrow{AX}$'. Pure mathematicians would not approve but this notation is so convenient we will use it anyway. It's all part of living dangerously!

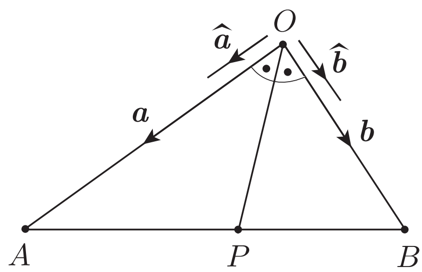

FIGURE 1.7 The bisector theorem: $AP/PB = OA/OB$.

Example 1.4 ***The bisector theorem***

In a triangle OAB, the bisector of the angle $A\widehat{O}B$ meets the line AB at the point P. Show that $AP/PB = OA/OB$.

Solution

Let the vertex O be the origin of vectors* and let the position vectors of the vertices A, B relative to O be $\boldsymbol{a}$, $\boldsymbol{b}$ as shown in Figure 1.7. The point with position vector $\boldsymbol{a}+\boldsymbol{b}$ does *not* lie the bisector OP in general since the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have different magnitudes a and b. However, by symmetry, the point with position vector $\widehat{\boldsymbol{a}}+\widehat{\boldsymbol{b}}$ *does* lie on the bisector and a general point X on the bisector has a position vector $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \lambda\left(\widehat{\boldsymbol{a}}+\widehat{\boldsymbol{b}}\right) = \lambda\left(\frac{\boldsymbol{a}}{a}+\frac{\boldsymbol{b}}{b}\right) = \lambda\left(\frac{b\,\boldsymbol{a}+a\,\boldsymbol{b}}{ab}\right) = \left(\frac{b\,\boldsymbol{a}+a\,\boldsymbol{b}}{K}\right),$$

where $K = ab/\lambda$ is a new constant. Now X will lie on the line AB if its position vector has the form $(\mu\boldsymbol{a}+\lambda\boldsymbol{b})/(\lambda+\mu)$, that is, if $K = a+b$. Hence the position vector $\boldsymbol{p}$ of P is

$$\boldsymbol{p} = \frac{b\,\boldsymbol{a}+a\,\boldsymbol{b}}{a+b}.$$

Moreover we see that P divides that line AB in the ratio $a : b$, that is, $AP/PB = OA/OB$ as required. ■

1.3 THE SCALAR PRODUCT $\boldsymbol{a}\cdot\boldsymbol{b}$

Definition 1.8 ***Scalar product*** *Suppose the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *have representations* $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$. *Then the* ***scalar product*** $\boldsymbol{a}\cdot\boldsymbol{b}$ *of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is defined by*

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta, \tag{1.3}$$

where θ *is the angle between* OA *and* OB. [Note that $\boldsymbol{a}\cdot\boldsymbol{b}$ is a **scalar** quantity.]

* One can always take a special point of the figure as origin. The penalty is that the symmetry of the labelling is lost.