Part A

General equilibrium theory: Getting acquainted

Chapter 1 begins to describe the concept of general equilibrium (simultaneous price-guided clearing of several goods markets) and gives some of the colorful history of its development over the nineteenth and twentieth centuries. Chapters 2 and 3 introduce two elegantly simple and insightful models of general equilibrium that are simple enough to present in elementary classes and rich enough to provide insights in advanced treatments:

- the Robinson Crusoe model, which emphasizes the interaction of the consumption and production sides of the economy and
- the Edgeworth box, which investigates bargaining and equilibrium in the exchange of commodities among consumers.

Chapters 4 and 5 include additional demonstrations:

- a characterization of the Pareto efficiency of general competitive equilibrium in a $2 \times 2 \times 2$ model (2 households, 2 outputs, 2 inputs) and
- a sample proof of existence of market general equilibrium, describing the structure of demand and supply functions needed to establish that prices can adjust so that markets can clear.

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Concept and history of general equilibrium theory

1.1 Partial and general equilibrium: Development of the field

The typical student's first exposure to an economic model consists of crossing supply and demand curves on the blackboard. They lead to a surprisingly definite result: Market prices are determined where the curves cross, at prices characterized by supply equaling demand. This is not merely a mathematical equality but a stationary position of a dynamic process – the price and quantity adjustments of the market. This is *partial* equilibrium, the adjustment of prices so that supply equals demand in a single market; the roles of other markets and prices are summarized by the qualification "other things being equal."

The conditions for finding a partial equilibrium are painfully simple. It is just that the supply and demand curves should cross, on the axis if nowhere else. Let p_k be the market price of good k, $S_k(p_k)$ be the supply function, and $D_k(p_k)$ the demand function. Equilibrium occurs at a price p_k^o where

$$S_k\left(p_k^o\right) = D_k\left(p_k^o\right), \text{ with } p_k^o \geq 0,$$

or

$$p_k^o = 0 \text{ if } S_k\left(p_k^o\right) > D_k\left(p_k^o\right).$$

In words, partial equilibrium occurs at a price so that supply equals demand, with the exception of free goods that may be in excess supply at an equilibrium price of zero. The notation here indicates that the market for good k is considered in isolation – only the price of good k is shown to enter the supply and demand functions for good k. This practice of isolating the market for each good separately is known as *partial* equilibrium analysis. The phrase "other things being equal" indicates that prices for all other goods are held fixed while considering the market for good k. The partial equilibrium is a powerfully simple technique, allowing us a successful first pass at issues of equilibrium, efficiency, and comparative statics (how prices may be expected to change with shifts in demand or supply).

What's wrong with partial equilibrium analysis? An example may help; let's try the U.S. market for SUVs (sports utility vehicles) in 2005 to 2008. Early in 2005, business prospects for the major U.S.-based automobile manufacturers (Chrysler, Ford, and General Motors) looked promising. SUV sales (a high-profit line of business) were robust. Then, midyear the firms reported deteriorating profits. The credit-worthiness ratings on their publicly traded debt were cut to junk bond levels. Their common stock share prices plunged. GM management was threatened with a hostile takeover. GM and Ford cut prices to clear out inventory, making the employee discount available to all customers. The news didn't get any better in 2006 or 2007, and then in 2008 it got worse. A billionaire investor threatened a takeover of Ford, then sold his stake in the company at an immense loss. GM and Ford sought and received loan guarantees from the U.S. federal government. From mid-2005 to mid-2008, an ownership share in GM fell in value by 75 percent; in Ford, it fell by 80 percent.[1]

What went wrong? Did Chrysler, Ford, and GM make an unusual mistake in 2004? Was there a new failure of management? Did a catastrophe threaten their manufacturing plants?

No. None of these adverse events took place. The SUV and automobile manufacturing situation were tranquil during the first part of 2005. The action was somewhere else: oil. The price of oil increased significantly in 2005–2008, hitting new all-time highs (in nominal dollar terms). Oil is used to make gasoline; SUVs use a lot of gasoline; demand for SUVs fell significantly. Automobile demand shifted to fuel-efficient cars, predominantly from non–U.S.-based manufacturers. The oil market trashed SUV sales and Chrysler, Ford, and GM profitability in 2005–2008.

Just looking at the market for SUVs wouldn't give you a handle on the Chrysler, Ford, and GM story for 2005–2008. You need to look at several markets at once: oil, gasoline, and SUVs. Interactions across markets are essential to forecasting and understanding economic activity. When we need to inquire into the interactions between markets, we relax the assumption of "other things being equal" and look at multiple markets simultaneously. Because there are distinctive interactions across markets (e.g., among the price of oil, the price of gasoline, and the demand for SUVs) it is important that the equilibrium concept include interactive simultaneous determination of equilibrium prices across markets. The concept can then represent a solution concept for the economy as a whole and not merely for a single market

[1] Of course, by 2009 the news was even worse. Chrysler and GM were reorganized in bankruptcy, with the U.S. federal government owning large portions of the companies (because no private investor would support their unprofitable operations). But a large portion of those failures reflects a credit crisis – a topic beyond the scope of this book.

artificially isolated. That is the concept of *general* economic equilibrium. General equilibrium for the economy consists of an array of prices for each good, where simultaneously supply equals demand for each good, while taking account of the interactions across markets. The prices of SUVs and of gasoline both adjust so that demand and supply of SUVs and of gasoline are each equated. That is *general* equilibrium; the equilibrium concept deals with all markets simultaneously and their interactions, rather than a single market in isolation. The economy is in general equilibrium when prices have fully adjusted so that supply equals demand in all markets. Let the goods be $k = 1, \ldots, N$. The demand and supply for good k will depend on the price of good k and on many other prices, so we denote them $D_k(p_1, p_2, \ldots, p_N)$ and $S_k(p_1, \ldots, p_N)$. Prices $p_1^o, p_2^o, \ldots, p_N^o$ are said to constitute general equilibrium prices if simultaneously each market is in equilibrium at the stated prices. That is, for all $k = 1, \ldots, N$,

$$D_k\left(p_1^o, p_2^o, \ldots, p_N^o\right) = S_k\left(p_1^o, \ldots, p_N^o\right), \; p_k^o \geq 0,$$

or

$$p_k^o = 0 \text{ for goods k such that } D_k\left(p_1^o, \ldots, p_N^o\right) < S_k\left(p_1^o, \ldots, p_N^o\right).$$

The distinction between general equilibrium and partial equilibrium is formally in the arguments of the functions D_k and S_k. All prices enter the supply and demand functions for good k, not merely the price of k. That's what makes this a *general* equilibrium. General equilibrium theory consists in studying these equilibria. In the process we will develop fundamental abstract models of the economy and an axiomatic method of analyzing them. Our most elementary model of general equilibrium, developed in Chapter 2, considers the market equilibrium for a Robinson Crusoe (one-person) economy. We investigate this example not because we actually expect a one-person economy to actively use a price system but because an economy so simple lets us easily analyze its efficient allocations and see directly the workings of the price system in all markets simultaneously. The balance of this book is designed to present the next step – a full mathematical model of the economy and its equilibrium price and allocation determination for all markets simultaneously.

General equilibrium analysis has proved fundamental in modern economics in describing the efficiency and stability of the market mechanism, in macroeconomic analysis, and in providing the logical foundations of microeconomics. One of the recurrent notions is to characterize the competitive market as *decentralized*. The idea of decentralization is that the complex interactive economic system is characterized by many independent decisionmakers who do not cooperate explicitly with one another. Nevertheless, their actions turn out to be consistent with one another because prices have adjusted for consistency and all the decision makers respond (separately and independently) to prices that are common to all. The remarkable result is that this lightly coordinated (decentralized) system nevertheless produces consistent and efficient allocation. This notion is investigated in the simple models of Chapters 2, 3, and 4 and more fully in Chapter 19. General equilibrium theory provides the basis for major innovations in modern economic theory and for the full mathematically rigorous confirmation of long-held traditional views in economics.

Why are economists interested in general equilibrium? The reason it is called *equilibrium* is that we expect there are forces in the economy, supply and demand, driving the system to this array of allocations and prices. That's where we expect the economy to end up or to move toward. Equilibrium is the descriptive and predictive principle for the market economist. Further, the desirable efficiency properties of a market economy depend on the economy being in general competitive equilibrium – or moving in that direction. The traditional major questions on equilibrium include:

- existence – the study of conditions under which there is a solution to the equations characterizing market clearing;
- uniqueness – whether there is only one family of prices that clears markets or there are multiple (or infinite) solutions to the market clearing problem;
- stability – whether a price formation mechanism that raises prices of goods in excess demand and reduces those in excess supply will converge to market clearing prices;
- efficiency – welfare economics, the effectiveness of the resource utilization implied at the equilibrium allocation; and
- bargaining – the relation of strategic bargaining solutions to passive price-taking equilibrium.

The treatment in this book, like that of the field, will concentrate on existence, efficiency, and bargaining in characterizing equilibrium.

We'll develop two separate ideas: (1) Efficient allocation of resources consists of technically efficient use of inputs to produce outputs and Pareto efficient allocation of consumption across households, and (2) competitive market equilibrium is a market clearing allocation guided by prices and firm and household optimization subject to market prices. Then we'll demonstrate a surprising result, the First Fundamental Theorem of Welfare Economics: The market equilibrium allocation is Pareto efficient.

Why is this surprising? The notion of market equilibrium is a very individualistic concept – firms and households each separately do the best they can without regard to others. Economists call this kind of decision making "decentralized." Pareto efficiency is a global concept. It takes account of all resources, tastes, and technologies available. When we calculate a Pareto-efficient allocation, the calculation takes all of these into account in an optimization. Economists call this viewpoint "centralized." The First Fundamental Theorem says that selfish, individually focused behavior in a market setting results in globally efficient use of resources. That's a surprise. The structure that allows this to happen is the market price system. Prices (of outputs and inputs) are visible to all in the market. They coordinate the individual activity. They apparently provide sufficient coordination that individually optimizing plans become globally efficient.

1.2 The role of mathematics

For several generations, economic theory and applications have become increasingly mathematical. The area of general equilibrium theory, necessarily abstract, has led in that movement, using the relatively abstract mathematical techniques of real analysis. The mathematics of N-dimensional space has turned out to be very suitable for modeling the interactions of N different markets for N goods produced by $\#F$ firms and consumed by $\#H$ households.

General equilibrium theory has been a particular leader in emphasizing the axiomatic method, stating assumptions clearly and definitely in mathematical form and deriving conclusions from them, making it explicitly an "if–then" exercise. Economics is an area where reason and intuition, assumptions and conclusions, tend to become confused and mix unpredictably. This is particularly true when considering the whole economy at once, rather than a single market. A disciplinary approach that emphasizes the logical development of ideas, clearly distinguishing between assumptions and conclusions, is then most appropriate. Much of what we know of the economy is based on simple, sometimes naive, intuition about individual economic units – firms and households. There is often broad agreement on the first principles governing their behavior, even when there is disagreement regarding conclusions and policy. This leads to a bottom-up approach stressing the construction of a model of the economy as a whole from agreed principles on firm and household behavior.

Professor Debreu (1986) tells us

> A consequence of the axiomatization of economic theory has been a greater clarity of expression, one of the most significant gains that it has achieved. To that effect, axiomatization does more than making assumptions and conclusions explicit and exposing the deductions linking them. The very definition of an economic concept is usually marred by a substantial margin of ambiguity. An axiomatized theory substitutes for that ambiguous concept a mathematical object that is subjected to definite rules of reasoning. Thus an axiomatic theorist succeeds in communicating the meaning he intends to give to a primitive concept because of the completely specified formal context in which he operates. The more developed this context is, the richer it is in theorems and in other primitive concepts, the smaller will be the margin of ambiguity in the intended interpretation.

The axiomatic method allows the theorist to develop general results: Wherever the assumptions are fulfilled, the conclusions will follow. That's the power of mathematical theory. Instead of working with examples and hoping that they generalize, the axiomatic approach states assumptions in general form and is rewarded with results that are generally applicable. These are "if . . . then" statements. If the assumptions are fulfilled, then the results follow.

Part of the underlying strategy of the theorist is a principle of *parsimony*; axioms should assume as little as possible (consistent with leading to useful conclusions), so that the applications can be as broad as possible. This approach has the colorful name "Ockham's Razor" after the medieval philosopher William of Ockham (1287–1347). In writing out a theorem, the assumptions are stated at the start, and a successful exposition will use – and need – all of the assumptions. Any assumption excessively strong or unneeded to achieve the conclusion represents an unnecessary restriction on the breadth of the result.

1.3 History of general equilibrium theory

Classical economists had a strong, if imprecise, notion of equilibrium. It represented the conditions that the economy centered on over time and returned to after a disturbance. The best-known statement of how equilibrium is achieved is more poetry than logic: Adam Smith's notion of an "invisible hand" guiding the market participants and the allocation mechanism. Nineteenth-century economists, including Ricardo, Mill, Marx, and Jevons, all recognized a notion of stable equilibrium tendencies in the economy and the importance of the interaction among markets (general equilibrium) without formalizing these notions mathematically.

The supply and demand diagram generally presented for partial equilibrium analysis is known as *Marshallian*, after the treatment of Alfred Marshall (1890), who popularized it in the English-speaking literature. Nevertheless, priority in the concept, its articulation, and mathematical presentation goes to Augustin Cournot (1838). That the modern attribution fails to give full credit to Cournot probably reflects the presentation of his ideas in two forms inaccessible to many readers: mathematics and French.

Cournot and other nineteenth-century writers clearly understood that partial equilibrium analysis presented a special case and that multiple market interactions were the appropriate generalization. They did not, however, formulate a full general equilibrium model. That exercise was first successfully undertaken by Leon Walras, a French economist at the School of Lausanne, Switzerland. His elegant comprehensive treatment appeared as *Elements of Pure Economics* (*Elements d'Economie Politique Pure*) in 1874. Walras set the problem and principal research agenda for all of twentieth-century mathematical general equilibrium theory. The

Walrasian model represented the first full recognition of the general equilibrium concept in the literature. It clearly stated that, for N commodities, there are N equations, $S_k(p_1, p_2, \ldots, p_N) = D_k(p_1, p_2, \ldots, p_N)$, in the N unknowns $p_n, n = 1, 2, \cdots, N$. Walras's approach to proving existence consisted in counting equations and unknowns to assure us that they were equal in number. If the equations were linear, independent, and otherwise unrestricted, this would constitute a sufficient condition for existence of a solution. But the equations will typically be nonlinear, and there are additional constraints on the system (in particular, nonnegativity requirements on quantities), so that equation counting will not typically ensure the existence of a solution.

F. Y. Edgeworth[2] presented the field with new concepts in bargaining and new tools to analyze them in *Mathematical Psychics* (1881). The modern elaboration of this inquiry takes place in Debreu and Scarf (1963) and is presented here in Chapters 21 and 22.

The modern period in general equilibrium theory starts amid the intellectual ferment and political instability of Vienna in the 1930s. The biweekly mathematics seminar chaired by the mathematician Karl Menger (son of the economist Carl Menger) included both the unemployed Hungarian mathematician Abraham Wald[3] and Karl Schlesinger, a wealthy Viennese banker and gifted amateur economist. To support Wald (who, in that period, was unemployable at the University of Vienna because he was Jewish), Menger arranged a private position for him with Schlesinger. Schlesinger introduced Wald to the problem of existence of general economic equilibrium. Wald presented mathematical proofs of existence of general equilibrium in a variety of models, each representing a special case of a general equilibrium system [see Wald (1934–35, 1936, 1951)]. With the deterioration of the political situation on the Continent, most of the seminar members subsequently emigrated to England and the United States, tragically with the exception of Schlesinger, who apparently committed suicide during the Nazi *Anschluss*.

In the early 1950s, three American authors, Kenneth Arrow, Gerard Debreu,[4] and Lionel McKenzie, entered the field. They worked at first separately and independently; then Arrow and Debreu worked in collaboration. The papers of Arrow and Debreu (1954) and McKenzie (1954) were presented to the 1952 meeting of

2 Edgeworth was by education a barrister (a lawyer specializing in advocacy in court), though he did not practice. His pioneering work of pure economic theory, *Mathematical Psychics*, was published before he held any academic position. He was appointed to a professorship at Kings College, London, in 1888, and in 1891 he assumed the prestigious Drummond Chair at Oxford. In addition to his enduring work in economics, Edgeworth is known for pioneering contributions to mathematical statistics.

3 Wald is often described inaccurately as Romanian, reflecting changes in the borders of the adjacent countries.

4 Debreu was then a French national on a fellowship at the Cowles Commission for Research in Economics at the University of Chicago. The allocation decision for one fellowship between two leading French economic theorists (Debreu and Marcel Boiteux) was based on the flip of a coin (administered by Maurice Allais). Dr. Marcel Boiteux was subsequently a leader in French economics and chief economist for Electricité de France.

the Econometric Society. It was the recognition by McKenzie and by Arrow and Debreu of the importance of using a fixed-point theorem that led to major progress in this area. The use of a fixed-point theorem for demonstrating the existence of an equilibrium [of a game] was pioneered by John Nash in 1950 (see Debreu, 1983). Additional contributions to the field in this period include Arrow (1951), restating the essential ideas of welfare economics in the language of general equilibrium theory, and Arrow (1953) extending the concept of commodity to include allocation under uncertainty (treated here in Chapter 20). The body of work was then summarized by Debreu (1959).

It is a commonplace in intermediate microeconomics that competitive price-taking behavior is most appropriate to a setting where there is a large number of buyers and sellers. Proving this result mathematically was the next major step in the progress of the general equilibrium theory. This is the elaboration of the Edgeworth bargaining model, culminating in the contribution of Debreu and Scarf (1963). They demonstrated Edgeworth's notion of equivalence, in a large economy, of price-taking equilibrium and the outcome of multilateral group and individual bargaining. The role of large numbers in a competitive economy is confirmed mathematically (Chapters 21 and 22 of this book). Arrow and Debreu received Nobel prizes in economics for their research in general equilibrium theory in 1972 and 1983, respectively. The class of general equilibrium economic models presented in this book is often called the Arrow-Debreu model.

The theory of general economic equilibrium remains an active, productive, demanding specialty of economic theory today. Each of the issues discussed in this chapter has gone through rich elaboration over the past several decades. Further research proceeds on allocation under uncertainty, general equilibrium models in industrial organization, monetary economics, and macroeconomics. Nevertheless, presenting the model as it was achieved in the mid-1960s allows a clear coherent and intuitive presentation with mathematics at the level of analysis in $\mathbf{R}^N$. This is essentially the treatment presented in most advanced textbooks in economic theory. The presentation of general equilibrium theory in this book is based on the model of Arrow and Debreu (1954). The treatment of allocative efficiency (welfare economics) is based on Arrow (1951). The notion of time reflects Hicks (1939). The treatment of uncertainty is based on Debreu (1959) and Arrow (1953). The treatment of bargaining and the core of a market economy is based on Debreu and Scarf (1963) and on Anderson (1978).

1.4 Bibliographic note

An excellent history of economic thought, including the formulation of the Edgeworth box and the general equilibrium theory of Walras, is available in Blaug

(1968). Walras's original – and still highly readable – exposition of the general equilibrium system is in Walras (1874). Weintraub (1983) describes the modern history of general equilibrium theory. Arrow (1989) provides a detailed discussion of the Viennese period. Arrow (1968) and Arrow and Hahn (1971) provide an analytic treatment of the history of thought. Duffie and Sonnenschein (1989) discusses in detail Kenneth Arrow's central role in development of the theory.