RANDOM DYNAMICAL SYSTEMS

This book provides an exposition of discrete time dynamic processes evolving over an infinite horizon. Chapter 1 reviews some mathematical results from the theory of deterministic dynamical systems, with particular emphasis on applications to economics. The theory of irreducible Markov processes, especially Markov chains, is surveyed in Chapter 2. Equilibrium and long-run stability of a dynamical system in which the law of motion is subject to random perturbations are the central theme of Chapters 3–5. A unified account of relatively recent results, exploiting splitting and contractions, that have found applications in many contexts is presented in detail. Chapter 6 explains how a random dynamical system may emerge from a class of dynamic programming problems. With examples and exercises, readers are guided from basic theory to the frontier of applied mathematical research.

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To Urmi, Deepta, and Aveek

Random Dynamical Systems

Theory and Applications

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Preface

The scope of this book is limited to the study of discrete time dynamic processes evolving over an infinite horizon. Its primary focus is on models with a one-period lag: "tomorrow" is determined by "today" through an exogenously given rule that is itself stationary or time-independent. A finite lag of arbitrary length may sometimes be incorporated in this scheme. In the deterministic case, the models belong to the broad mathematical class, known as dynamical systems, discussed in Chapter 1, with particular emphasis on those arising in economics. In the presence of random perturbations, the processes are random dynamical systems whose long-term stability is our main quest. These occupy a central place in the theory of discrete time stochastic processes.

Aside from the appearance of many examples from economics, there is a significant distinction between the presentation in this book and that found in standard texts on Markov processes. Following the exposition in Chapter 2 of the basic theory of irreducible processes, especially Markov chains, much of Chapters 3–5 deals with the problem of stability of random dynamical systems which may not, in general, be irreducible. The latter models arise, for example, if the random perturbation is limited to a finite or countable number of choices. Quite a bit of this theory is of relatively recent origin and appears especially relevant to economics because of underlying structures of monotonicity or contraction. But it is useful in other contexts as well.

In view of our restriction to discrete time frameworks, we have not touched upon powerful techniques involving deterministic and stochastic differential equations or calculus of variations that have led to significant advances in many disciplines, including economics and finance.

It is not possible to rely on the economic data to sift through various possibilities and to compute estimates with the degrees of precision that natural or biological scientists can often achieve through controlled experiments. We duly recognize that there are obvious limits to the lessons that formal models with exogenously specified laws of motion can offer.

The first chapter of the book presents a treatment of deterministic dynamical systems. It has been used in a course on dynamic models in economics, addressed to advanced undergraduate students at Cornell. Supplemented by appropriate references, it can also be part of a graduate course on dynamic economics. It requires a good background in calculus and real analysis.

Chapters 2–6 have been used as the core material in a graduate course at Cornell on Markov processes and their applications to economics. An alternative is to use Chapters 1–3 and 5 to introduce models of intertemporal optimization/equilibrium and the role of uncertainty. Complements and Details make it easier for the researchers to follow up on some of the themes in the text.

In addition to numerous examples illustrating the theory, many exercises are included for pedagogic purposes. Some of the exercises are numbered and set aside in paragraphs, and a few appear at the end of some chapters. But quite a few exercises are simply marked as (Exercise), in the body of a proof or an argument, indicating that a relatively minor step in reasoning needs to be formally completed.

Given the extensive use of the techniques that we review, we are unable to provide a bibliography that can do justice to researchers in many disciplines. We have cited several well-known monographs, texts, and review articles which, in turn, have extended lists of references for curious readers.

The quote attributed to Toni Morrison in Chapter 1 is available on the Internet from Simpson's Contemporary Quotations, compiled by J. B. Simpson.

The quote from Shizuo Kakutani in Chapter 2 is available on the Internet at www.uml.edu/Dept/Math/alumni/tangents/tangents_Fall2004/MathInTheNews.htm. Endnote 1 of the document describes it as "a joke by Shizuo Kakutani at a UCLA colloquium talk as attributed in Rick Durrett's book *Probability*: *Theory and Examples*." The other quote in this chapter is adapted from Bibhuti Bandyopadhyay's original masterpiece in Bengali.

The quote from Gerard Debreu in Chapter 4 appeared in his article in *American Economic Review* (Vol. 81, 1991, pp. 1–7).

The quote from Patrick Henry in Chapter 5 is from Bartlett's Quotations (no. 4598), available on the Internet.

The quote attributed to Freeman J. Dyson in the same chapter appeared in the circulated abstract of his Nordlander Lecture ("The Predictable and the Unpredictable: How to Tell the Difference") at Cornell University on October 21, 2004.

The quote from Kenneth Arrow at the beginning of Chapter 6 appears in Chapter 2 of his classic *Essays in the Theory of Risk-Bearing.*

Other quotes are from sources cited in the text.

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Two collections of published articles have played an important role in our exposition: a symposium on *Chaotic Dynamical Systems* (edited by Mukul Majumdar) and a symposium on *Dynamical Systems Subject to Random Shocks* (edited by Rabi Bhattacharya and Mukul Majumdar) that appeared in *Economic Theory* (the first in Vol. 4, 1995, and the second in Vol. 23, 2004). We acknowledge the enthusiastic support of Professor C. D. Aliprantis in this context.

Finally, thanks are due to Scott Parris, who initiated the project.

Notation

$\mathbb{Z}$	set of *all* integers.
$\mathbb{Z}_+(\mathbb{Z}_{++})$	set of all *nonnegative* (*positive*) integers.
$\mathbb{R}$	set of *all* real numbers.
$\mathbb{R}_+(\mathbb{R}_{++})$	set of all *nonnegative* (*positive*) real numbers.
$\mathbb{R}^\ell$	set of all ℓ-vectors.
$\mathbf{x} = (x_i) = (x_1, \ldots, x_\ell)$	an element of $\mathbb{R}^\ell$.
$\mathbf{x} \geq 0$	$x_i \geq 0$ for $i = 1, 2, \ldots, \ell$; [$\mathbf{x}$ is *nonnegative*].
$\mathbf{x} > 0$	$x_i \geq 0$ for all i; $x_i > 0$ for some i; [$\mathbf{x}$ is *positive*].
$\mathbf{x} \gg 0$	$x_i > 0$ for all i; [$\mathbf{x}$ is *strictly positive*].
$(S, \mathcal{S})$	a measurable space [when S is a metric space, $\mathcal{S} = \mathcal{B}(S)$ is the Borel sigmafield unless otherwise specified].