# 1

# Dynamical Systems

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.

Robert M. May

There is nothing more to say – except why. But since why is difficult to handle, one must take refuge in how.

Toni Morrison

## 1.1 Introduction

There is a rich literature on discrete time models in many disciplines – including economics – in which dynamic processes are described formally by first-order difference equations (see (2.1)). Studies of dynamic properties of such equations usually involve an appropriate definition of a steady state (viewed as a dynamic equilibrium) and conditions that guarantee its existence and local or global stability. Also of importance, particularly in economics following the lead of Samuelson (1947), have been the problems of comparative statics and dynamics: a systematic analysis of how the steady states or trajectories respond to changes in some parameter that affects the law of motion. While the dynamic properties of linear systems (see (4.1)) have long been well understood, relatively recent studies have emphasized that "the very simplest" nonlinear difference equations can exhibit "a wide spectrum of qualitative behavior," from stable steady states, "through cascades of stable cycles, to a regime in which the behavior (although fully deterministic) is in many respects chaotic or indistinguishable from the sample functions of a random process" (May 1976, p. 459). This chapter is not intended to be a comprehensive review of the properties of complex dynamical systems, the study of which has benefited from a collaboration between the more "abstract" qualitative analysis of difference and differential equations, and a careful exploration of "concrete" examples through increasingly sophisticated computer experiments. It does recall some of the basic results on dynamical systems, and draws upon a variety of examples from economics (see Complements and Details).

There is by now a plethora of definitions of "chaotic" or "complex" behavior, and we touch upon a few properties of chaotic systems in Sections 1.2 and 1.3. However, the map (2.3) and, more generally, the quadratic family discussed in Section 1.7 provide a convenient framework for understanding many of the definitions, developing intuition and achieving generalizations (see Complements and Details). It has been stressed that the qualitative behavior of the solution to Equation (2.5) depends crucially on the initial condition. Trajectories emanating from initial points that are very close may display radically different properties. This may mean that small changes in the initial condition "lead to predictions so different, after a while, that prediction becomes in effect useless" (Ruelle 1991, p. 47). Even within the quadratic family, complexities are not "knife-edge," "abnormal," or "rare" possibilities. These observations are particularly relevant for models in social sciences, in which there are obvious limits to gathering data to identify the initial condition, and avoiding computational errors at various stages.

In Section 1.2 we collect some basic results on the existence of fixed points and their stability properties. Of fundamental importance is the contraction mapping theorem (Theorem 2.1) used repeatedly in subsequent chapters. Section 1.3 introduces complex dynamical systems, and the central result is the Li–Yorke theorem (Theorem 3.1). In Section 1.4 we briefly touch upon linear difference equations. In Section 1.5 we explore in detail dynamical systems in which the state space is $\mathbb{R}_+$, the set of nonnegative reals, and the law of motion $\alpha$ is an increasing function. Proposition 5.1 is widely used in economics and biology: it identifies a class of dynamical systems in which all trajectories (emanating from initial $x$ in $\mathbb{R}_{++}$) converge to a unique fixed point. In contrast, Section 1.6 provides examples in which the long-run behavior depends on initial conditions. In the development of complex dynamical systems, the "quadratic family" of laws of motion (see (7.11)) has played a distinguished role. After a review of some results on this family in Section 1.7, we turn to examples of dynamical systems from economics and biology.

We have selected some descriptive models, some models of optimization with a single decision maker, a dynamic game theoretic model, and an example of intertemporal equilibrium with overlapping generations. An interesting lesson that emerges is that variations of some well-known models that generate monotone behavior lead to dynamical systems exhibiting Li–Yorke chaos, or even to systems with the quadratic family as possible laws of motion.

## 1.2 Basic Definitions: Fixed and Periodic Points

We begin with some formal definitions. A dynamical system is described by a pair $(S, \alpha)$ where $S$ is a nonempty set (called the *state space*) and $\alpha$ is a function (called the *law of motion*) from $S$ into $S$. Thus, if $x_t$ is the state of the system in period $t$, then

$$x_{t+1} = \alpha(x_t) \tag{2.1}$$

is the state of the system in period $t + 1$.

In this chapter we always assume that the state space $S$ *is a (nonempty) metric space (the metric is denoted by d).* As examples of (2.1), take $S$ to be the set $\mathbb{R}$ of real numbers, and define

$$\alpha(x) = ax + b, \tag{2.2}$$

where $a$ and $b$ are real numbers.

Another example is provided by $S = [0, 1]$ and

$$\alpha(x) = 4x(1 - x). \tag{2.3}$$

Here in (2.3), $d(x, y) \equiv |x - y|$.

The evolution of the dynamical system $(\mathbb{R}, \alpha)$ where $\alpha$ is defined by (2.2) is described by the difference equation

$$x_{t+1} = ax_t + b. \tag{2.4}$$

Similarly, the dynamical system $([0, 1], \alpha)$ where $\alpha$ is defined by (2.3) is described by the difference equation

$$x_{t+1} = 4x_t(1 - x_t). \tag{2.5}$$

Once the initial state $x$ (i.e., the state in period 0) is specified, we write $\alpha^0(x) \equiv x$, $\alpha^1(x) = \alpha(x)$, and for every positive integer $j \geq 1$,

$$\alpha^{j+1}(x) = \alpha(\alpha^j(x)). \tag{2.6}$$

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We refer to $\alpha^j$ as the *jth iterate* of $\alpha$. For any initial $x$, the *trajectory* from $x$ is the sequence $\tau(x) = \{\alpha^j(x)_{j=0}^{\infty}\}$. The *orbit* from $x$ is the set $\gamma(x) = \{y\colon y = \alpha^j(x) \text{ for some } j \geq 0\}$. The limit set $w(x)$ of a trajectory $\tau(x)$ is defined as

$$w(x) = \bigcap_{j=1}^{\infty} \overline{[\tau(\alpha^j(x))]}, \tag{2.7}$$

where $\bar{A}$ is the closure of $A$.

Fixed and periodic points formally capture the intuitive idea of a *stationary* state or an *equilibrium* of a dynamical system. In his *Foundations*, Samuelson (1947, p. 313) noted that "*Stationary* is a descriptive term characterizing the behavior of an economic variable over time; it usually implies constancy, but is occasionally generalized to include behavior periodically repetitive over time."

A point $x \in S$ is a *fixed point* if $x = \alpha(x)$. A point $x \in S$ is a *periodic point* of *period* $k \geq 2$ if $\alpha^k(x) = x$ and $\alpha^j(x) \neq x$ for $1 \leq j < k$. Thus, to prove that $x$ is a periodic point of period, say, 3, one must prove that $x$ is a fixed point of $\alpha^3$ and that it is *not* a fixed point of $\alpha$ and $\alpha^2$. *Some writers consider a fixed point as a periodic point of period* 1.

Denote the set of all periodic points of $S$ by $\wp(S)$. We write $\aleph(S)$ to denote the set of nonperiodic points.

We now note some useful results on the existence of fixed points of $\alpha$.

**Proposition 2.1** *Let $S = \mathbb{R}$ and $\alpha$ be continuous. If there is a (nondegenerate) closed interval $I = [a, b]$ such that (i) $\alpha(I) \subset I$ or (ii) $\alpha(I) \supset I$, then there is a fixed point of $\alpha$ in $I$.*

*Proof.*

(i) If $\alpha(I) \subset I$, then $\alpha(a) \geq a$ and $\alpha(b) \leq b$. If $\alpha(a) = a$ or $\alpha(b) = b$, the conclusion is immediate. Otherwise, $\alpha(a) > a$ and $\alpha(b) < b$. This means that the function $\beta(x) = \alpha(x) - x$ is positive at $a$ and negative at $b$. Using the intermediate value theorem, $\beta(x^*) = 0$ for some $x^*$ in $(a, b)$. Then $\alpha(x^*) = x^*$.

(ii) By the Weierstrass theorem, there are points $x_m$ and $x_M$ in $I$ such that $\alpha(x_m) \leq \alpha(x) \leq \alpha(x_M)$ for all $x$ in $I$. Write $\alpha(x_m) = m$ and $\alpha(x_M) = M$. Then, by the intermediate value theorem, $\alpha(I) = [m, M]$.

Since $\alpha(I) \supset I$, $m \leq a \leq b \leq M$. In other words,

$$\alpha(x_m) = m \leq a \leq x_m,$$

and

$$\alpha(x_M) = M \geq b \geq x_M.$$

The proof can now be completed by an argument similar to that in case (i). ∎

**Remark 2.1** Let $S = [a, b]$ and $\alpha$ be a continuous function from $S$ into $S$. Suppose that for all $x$ in $(a, b)$ the derivative $\alpha'(x)$ exists and $|\alpha'(x)| < 1$. Then $\alpha$ has a unique fixed point in $S$.

**Proposition 2.2** *Let $S$ be a nonempty compact convex subset of $\mathbb{R}^\ell$, and $\alpha$ be continuous. Then there is a fixed point of $\alpha$.*

A function $\alpha : S \to S$ is a *uniformly strict contraction* if there is some $\mathcal{C}$, $0 < \mathcal{C} < 1$, such that for all $x, y \in X$, $x \neq y$, one has

$$d(\alpha(x),\ \alpha(y)) < \mathcal{C}d(x, y). \tag{2.8}$$

If $d(\alpha(x), \alpha(y)) < d(x, y)$ for $x \neq y$, we say that $\alpha$ is a *strict contraction*. If only

$$d(\alpha(x), \alpha(y)) \leqq d(x, y),$$

we say that $\alpha$ is a *contraction*.

If $\alpha$ is a contraction, $\alpha$ is continuous on $S$.

In this book, the following fundamental theorem is used many times:

**Theorem 2.1** *Let $(S, d)$ be a nonempty complete metric space and $\alpha : S \to S$ be a uniformly strict contraction. Then $\alpha$ has a unique fixed point $x^* \in S$. Moreover, for any $x$ in $S$, the trajectory $\tau(x) = \{\alpha^j(x)_{j=0}^{\infty}\}$ converges to $x^*$.*

*Proof.* Choose an arbitrary $x \in S$. Consider the trajectory $\tau(x) = (x_t)$ from $x$, where

$$x_{t+1} = \alpha(x_t). \tag{2.9}$$

Note that $d(x_2, x_1) = d(\alpha(x_1), \alpha(x)) < \mathcal{C}d(x_1, x)$ for some $\mathcal{C} \in (0, 1)$; hence, for any $t \geq 1$,

$$d(x_{t+1}, x_t) < \mathcal{C}^t d(x_1, x). \tag{2.10}$$

We note that

$$\begin{aligned} d(x_{t+2}, x_t) &\leq d(x_{t+2}, x_{t+1}) + d(x_{t+1}, x_t) \\ &< \mathcal{C}^{t+1} d(x_1, x) + \mathcal{C}^t d(x_1, x) \\ &= \mathcal{C}^t (1 + \mathcal{C}) d(x_1, x). \end{aligned}$$

It follows that for any integer $k \geq 1$,

$$d(x_{t+k}, x_t) < [\mathcal{C}^t / (1 - \mathcal{C})] d(x_1, x),$$

and this implies that $(x_t)$ is a Cauchy sequence. Since $S$ is assumed to be complete, $\text{limit}_{t\to\infty} x_t = x^*$ exists. By continuity of $\alpha$, and (2.9),

$$\alpha(x^*) = x^*.$$

If there are two distinct fixed points $x^*$ and $x^{**}$ of $\alpha$, we see that there is a contradiction:

$$0 < d(x^*, x^{**}) = d(\alpha(x^*), \alpha(x^{**})) < \mathcal{C}d(x^*, x^{**}), \tag{2.11}$$

where $0 < \mathcal{C} < 1$. ■

**Remark 2.2** For applications of this fundamental result, it is important to reflect upon the following:

(i) for any $x \in S$, $d(\alpha^n(x), x^*) \leq C^n (1 - C)^{-1} d(\alpha(x), x))$,
(ii) for any $x \in S$, $d(x, x^*) \leq (1 - C)^{-1} d(\alpha(x), x)$.

**Theorem 2.2** *Let $S$ be a nonempty complete metric space and $\alpha : S \to S$ be such that $\alpha^k$ is a uniformly strict contraction for some integer $k > 1$. Then $\alpha$ has a unique fixed point $x^* \in S$.*

*Proof.* Let $x^*$ be the unique fixed point of $\alpha^k$. Then

$$\alpha^k(\alpha(x^*)) = \alpha(\alpha^k(x^*)) = \alpha(x^*)$$

Hence $\alpha(x^*)$ is also a fixed point of $\alpha^k$. By uniqueness, $\alpha(x^*) = x^*$. This means that $x^*$ is a fixed point of $\alpha$. But *any* fixed point of $\alpha$ is a fixed point of $\alpha^k$. Hence $x^*$ is the unique fixed point of $\alpha$. ■

**Theorem 2.3** *Let $S$ be a nonempty compact metric space and $\alpha : S \to S$ be a strict contraction. Then $\alpha$ has a unique fixed point.*

*Proof.* Since $d(\alpha(x), x)$ is continuous and $S$ is compact, there is an $x^* \in S$ such that

$$d(\alpha(x^*), x^*) = \inf_{x \in S} d(\alpha(x), x). \tag{2.12}$$

Then $\alpha(x^*) = x^*$, otherwise

$$d(\alpha^2(x^*), \alpha(x^*)) < d(\alpha(x^*), x^*),$$

contradicting (2.12). ■

**Exercise 2.1**

(a) Let $S = [0, 1]$, and consider the map $\alpha : S \to S$ defined by

$$\alpha(x) = x - \frac{x^2}{2}.$$

Show that $\alpha$ is a strict contraction, but *not* a uniformly strict contraction. Analyze the behavior of trajectories $\tau(x)$ from $x \in S$.

(b) Let $S = \mathbb{R}$, and consider the map $\alpha : S \to S$ defined by

$$\alpha(x) = [x + (x^2 + 1)^{1/2}]/2.$$

Show that $\alpha(x)$ is a strict contraction, but does not have a fixed point. ■

A fixed point $x^*$ of $\alpha$ is (locally) *attracting* or (locally) *stable* if there is an open set $U$ containing $x^*$ such that for all $x \in U$, the trajectory $\tau(x)$ from $x$ converges to $x^*$.

We shall often drop the caveat "local": note that *local attraction* or *local stability* is to be distinguished from the property of *global stability* of a dynamical system: $(S, \alpha)$ is *globally stable* if for all $x \in S$, the trajectory $\tau(x)$ converges to the unique fixed point $x^*$. Theorem 2.1 deals with *global stability*.

A fixed point $x^*$ of $\alpha$ is *repelling* if there is an open set $U$ containing $x^*$ such that for any $x \in U$, $x \neq x^*$, there is some $k \geq 1$, $\alpha^k(x) \notin U$.

Consider a dynamical system $(S, \alpha)$ where $S$ is a (nondegenerate) closed interval $[a, b]$ and $\alpha$ is continuous on $[a, b]$. Suppose that $\alpha$ is also continuously differentiable on $(a, b)$. A fixed point $x^* \in (a, b)$ is *hyperbolic* if $|\alpha'(x^*)| \neq 1$.

**Proposition 2.3** *Let $S = [a, b]$ and $\alpha$ be continuous on $[a, b]$ and continuously differentiable on $(a, b)$. Let $x^* \in (a, b)$ be a hyperbolic fixed point of $\alpha$.*

*(a) If $|\alpha'(x^*)| < 1$, then $x^*$ is locally stable.*
*(b) If $|\alpha'(x^*)| > 1$, then $x^*$ is repelling.*

*Proof.*

(a) There is some $u > 0$ such that $|\alpha'(x)| < \mathbf{m} < 1$ for all $x$ in $I = [x^* - u, x^* + u]$. By the mean value theorem, if $x \in I$,

$$|\alpha(x) - x^*| = |\alpha(x) - \alpha(x^*)| \leq \mathbf{m}|x - x^*| < \mathbf{m}u < u.$$

Hence, $\alpha$ maps $I$ into $I$ and, again, by the mean value theorem, is a uniformly strict contraction on $I$. The result follows from Theorem 2.1.

(b) this is left as an exercise. ■

We can define "a hyperbolic periodic point of period $k$" and define (locally) attracting and repelling periodic points accordingly.

Let $x_0$ be a periodic point of period 2 and $x_1 = \alpha(x_0)$. By definition $x_0 = \alpha(x_1) = \alpha^2(x_0)$ and $x_1 = \alpha(x_0) = \alpha^2(x_1)$. Now if $\alpha$ is differentiable, by the chain rule,

$$[\alpha^2(x_0)]' = \alpha'(x_1)\alpha'(x_0).$$

More generally, suppose that $x_0$ is a periodic point of period $k$ and its orbit is denoted by $\{x_0, x_1, \ldots, x_{k-1}\}$. Then,

$$[\alpha^k(x_0)]' = \alpha'(x_{k-1}) \cdots \alpha'(x_0).$$

It follows that

$$[\alpha^k(x_0)]' = [\alpha^k(x_1)]' \cdots [\alpha^k(x_{k-1})]'.$$

We can now extend Proposition 2.3 appropriately.

While the contraction property of $\alpha$ ensures that, independent of the initial condition, the trajectories enter any neighborhood of the fixed point, there are examples of simple nonlinear dynamical systems in which trajectories "wander around" the state space. We shall examine this feature more formally in Section 1.3.

**Example 2.1** Let $S = \mathbb{R}$, $\alpha(x) = x^2$. Clearly, the only fixed points of $\alpha$ are 0, 1. More generally, keeping $S = \mathbb{R}$, consider the family of dynamical systems $\alpha_\theta(x) = x^2 + \theta$, where $\theta$ is a real number. For $\theta > 1/4$, $\alpha_\theta$ does not have any fixed point; for $\theta = 1/4$, $\alpha_\theta$ has a unique fixed point $x = 1/2$; for $\theta < 1/4$, $\alpha_\theta$ has a pair of fixed points.

When $\theta = -1$, the fixed points of the map $\alpha_{(-1)}(x) = x^2 - 1$ are $[1 + \sqrt{5}]/2$ and $[1 - \sqrt{5}]/2$. Now $\alpha_{(-1)}(0) = -1$; $\alpha_{(-1)}(-1) = 0$. Hence, both 0 and $-1$ are periodic points of period 2 of $\alpha_{(-1)}$. It follows that:

$$\tau(0) = (0, -1, 0, -1, \ldots), \quad \tau(-1) = (-1, 0, -1, 0, \ldots),$$
$$\gamma(-1) = \{-1, 0\}, \gamma(0) = \{0, -1\}.$$

Since

$$\alpha^2_{(-1)}(x) = x^4 - 2x^2,$$

we see that (i) $\alpha^2_{(-1)}$ has four fixed points: the fixed points of $\alpha_{(-1)}$, and $0, -1$; (ii) the derivative of $\alpha^2_{(-1)}$ with respect to $x$, denoted by $[\alpha^2_{(-1)}(x)]'$, is given by

$$[\alpha^2_{(-1)}(x)]' = 4x^3 - 4x.$$

Now, $[\alpha^2_{(-1)}(x)]'_{x=0} = [\alpha^2_{(-1)}(x)]'_{x=-1} = 0$. Hence, both 0 and $-1$ are attracting fixed points of $\alpha^2$. ■

**Example 2.2** Let $S = [0, 1]$. Consider the "tent map" defined by

$$\alpha(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2(1 - x) & \text{for } x \in [1/2, 1]. \end{cases}$$

Note that $\alpha$ has two fixed points "0" and "2/3." It is tedious to write out the functional form of $\alpha^2$:

$$\alpha^2(x) = \begin{cases} 4x & \text{for } x \in [0, 1/4] \\ 2(1 - 2x) & \text{for } x \in [1/4, 1/2] \\ 2(2x - 1) & \text{for } x \in [1/2, 3/4] \\ 4(1 - x) & \text{for } x \in [3/4, 1]. \end{cases}$$

Verify the following:

(i) "2/5" and "4/5" are periodic points of period 2.

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(ii) "2/9," "4/9," "8/9" are periodic points of period 3. It follows from a well-known result (see Theorem 3.1) that there are periodic points of *all* periods.

By using the graphs, if necessary, verify that the fixed and periodic points of the tent map are repelling. ■

**Example 2.3** In many applications to economics and biology, the state space $S$ is the set of all nonnegative reals, $S = \mathbb{R}_+$. The law of motion $\alpha : S \to S$ has the special form

$$\alpha(x) = x\beta(x), \tag{2.11'}$$

where $\beta(0) \geq 0$, $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous (and often has additional properties). Now, the fixed points $\hat{x}$ of $\alpha$ must satisfy

$$\alpha(\hat{x}) = \hat{x}$$

or

$$\hat{x}[1 - \beta(\hat{x})] = 0.$$

The fixed point $\hat{x} = 0$ may have a special significance in a particular context (e.g., extinction of a natural resource). Some examples of $\alpha$ satisfying (2.11′) are

| | | |
|---|---|---|
| (Verhulst 1845) | $\alpha(x) = \dfrac{\theta_1 x}{x + \theta_2}$, | $\theta_1 > 0, \theta_2 > 0.$ |
| (Hassell 1975) | $\alpha(x) = \theta_1 x(1 + x)^{-\theta_2}$, | $\theta_1 > 0, \theta_2 > 0.$ |
| (Ricker 1954) | $\alpha(x) = \theta_1 x e^{-\theta_2 x}$, | $\theta_1 > 0, \theta_2 > 0.$ |

Here $\theta_1, \theta_2$ are interpreted as exogenous parameters that influence the law of motion $\alpha$.

Assume that $\beta(x)$ is differentiable at $x \geq 0$. Then,

$$\alpha'(x) = \beta(x) + x\beta'(x).$$

Hence,

$$\alpha'(0) = \beta(0).$$

For each of the special maps, the existence of a fixed point $\hat{x} \neq 0$ and the local stability properties depend on the values of the parameters $\theta_1, \theta_2$. We shall now elaborate on this point.