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R. Daniel Mauldin and Mariusz Urbanski
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Introduction

The geometric and dynamic theory of the limit set generated by the iteration of finitely many similarity maps satisfying the open set condition has been well developed for some time now. Over the past several years, the authors have in turn developed a technically more complicated geometric and dynamic theory of the limit set generated by the iteration of infinitely many uniformly contracting conformal maps, a (hyperbolic) conformal iterated function system. This theory allows one to analyze many more limit sets, for example sets of continued fractions with restricted entries. We recall and extend this theory in the later chapters. The main focus of this book is the exploration of the geometric and dynamic properties of a far reaching generalization of a conformal iterated function system called a graph directed Markov system (GDMS). These systems are very robust in that they apply to many settings that do not fit into the scheme of conformal iterated systems. While the basic theory is laid out here and we touch on many natural questions arising in its context, we emphasize that there are many issues and current research topics which we do not cover: for examples, the detailed analysis of the structure of harmonic measures of limit sets provided in [UZd], the examination of the doubling property of conformal measures performed in [MU6], the extensive study of generalized polynomial like mappings (see [U7] and [SU]), the multifractal analysis of geometrically finite Kleinian groups (see [KS]), and the connection to quantization dimension from engineering (see [LM] and [GL]). There are many research problems in this active area that remain unsolved.

Our book is organized as follows. In the very short first chapter we describe the basic setting for GDMSs. Essentially, one iterates a family of uniformly contracting maps which are indexed by the directed edges in a multigraph which may have infinitely many vertices and infinitely

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many edges. (This includes the finite graph directed systems of similarities introduced in [MW2, EM] and expounded by Edgar in [E].) One generates points in the limit set by performing an infinite directed walk through the graph. This leads to a natural map from the coding space or space of infinite walks through the graph to the points of the limit set.

Chapter 2 forms a self-contained unit and can be read independently of the rest of the book. Here we develop the symbolic dynamics and thermodynamic formalism for subshifts of finite type with infinitely many symbols. Thus, we are given an incidence matrix A and we consider the space E of all infinite sequences of symbols such that A has value 1 on all pairs of consecutive terms and the shift map on E. This formalism includes the action on the coding space described in the first chapter. One of our main goals is to develop the theory and properties of various measures and functions on the symbol space. To a function f on the symbol space is associated its topological pressure with respect to the shift map. Of course, this is a standard thing to do, but care must be taken because we have infinitely many symbols and the space E is not compact. In the first section we determine some conditions under which the topological pressure may be approximated by the more usual pressures when the system is restricted to subsystems on finitely many symbols. Next, we determine conditions under which there exists an invariant Gibbs state for the functions f. We present some results of ourselves and of Sarig which state that for a reasonable class of functions f which we call acceptable, there is an invariant Gibbs state for f if and only if the matrix A is finitely irreducible. We also determine some conditions under which f has a unique invariant equilibrium state. In the third section we develop the properties of the transfer or Perron–Frobenius operator associated to f. In order to fully analyze our system, we provide some results from functional analysis in the fourth section. We prove an exponential decay of correlations, a central limit theorem and a generalized law of the iterated logarithm in section 5. In section 6, we vary the function f with a complex parameter t. We show that various operators are then holomorphic, which implies under appropriate assumptions real analyticity of the pressure function. In section 7, we show that for certain functions f, the associated conjugate Perron–Frobenius operator has a Borel probability measure as an eigenmeasure. This allows us to conclude that if A is finitely primitive then there is a Gibbs state for f.

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In Chapter 3, by using the tools developed in Chapter 2, we put the thermodynamic formalism for infinite subshifts of finite type into the context of GDMSs. We begin with F, a Hölder family of weight functions associated to the GDMS. By using an associated topological pressure and Perron–Frobenius operator, we determine conditions under which there is an F-conformal measure. This very general definition of F-conformal measure not only generalizes the usual notion of conformal measure, but forms a basic tool for later use in obtaining the geometric properties of the limit sets.

In Chapter 4 we deal in detail with geometric and fractal properties of conformal GDMSs. We begin by proving various kinds of distortion properties of conformal maps in $\mathbb{R}^d$ with $d \geq 2$. We then deal in this chapter with the various basic notions of dimension for the limit set: Hausdorff, upper and lower Minkowski or box (ball) counting dimension and packing dimension and the corresponding Hausdorff and packing measures. We deal with the Hausdorff and packing dimension of various natural measures supported on the limit set and some geometric properties of the limit set, e.g. porosity. Finally, we obtain the multifractal analysis of various conformal measures supported on the limit set. We emphasize that it is in this chapter that we must transfer many results from the abstract coding space to the limit set. As a point may have more than one code, this leads to several delicate issues in geometric measure theory. Therefore, the roles of distortion properties of our system of maps and the geometric properties of our seed sets, e.g. a relatively reasonable boundary, "the cone condition," play a crucial role in obtaining the necessary estimates of our analysis.

Chapter 5 is devoted to various illustrative examples including Kleinian groups of Schottky type, expanding repellers and a number of one-dimensional systems with prescribed geometric features.

In Chapter 6 we start to present the special case of a GDMS, a conformal iterated function system (CIFS). We study the real-analytic extension of the Radon-Nikodym derivative of an invariant measure with respect to a conformal measure, the classical example being Gauss' measure for the shift map on continued fractions. We estimate the rate at which the Hausdorff dimension of the limit sets generated by the finite subsystems of a CIFS approximates the dimension of the limit set. We determine conditions under which the limit set is uniformly perfect. In Section 4 of this chapter we begin the discussion of geometric rigidity by dealing with the limits set of a CIFS whose closure is connected. In essence the rigidity in this section means that in case $d \geq 3$ either the

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Hausdorff dimension of the limit set is larger than 1 or else in case $d \geq 3$ the limit set is a subset of circle or line and in case $d = 2$ it is a subset of an analytic Jordan curve. In the next section we improve on this rigidity by showing that if $d \geq 3$ then essentially either the Hausdorff dimension of the limit set J exceeds the topological dimension k of the closure of J or else the closure of J is a proper compact subset of either a geometric sphere or an affine subspace of dimension k.

In Chapter 7 we deal with dynamical rigidity stemming from the work of Sullivan (see [Su3]) on conformal expanding repellers in the complex plane. We ask the fundamental question when two topologically conjugate infinite iterated function systems are conjugate in a smoother fashion. The answer is that such conjugacy extends to a conformal conjugacy on some neighborhoods of limit sets if and only if it is Lipschitz continuous. This turns out to equivalently mean that this conjugacy exchanges measure classes of appropriate conformal measures or that the multipliers of corresponding fixed points of all compositions of our generators coincide.

In Chapter 8 we study PIFS, parabolic iterated function systems. These are systems that are almost conformal except that we allow finitely many of the maps to have a parabolic or neutral fixed point instead of being uniformly contracting. A prime example of such a system is given by the system of three conformal maps in the plane whose limit set is the residual set in Apollonian packing. We analyze these systems by showing how to associate a conformal iterated function system to the parabolic system and how the properties of the parabolic limit set and measures may be derived from this associated conformal system. It is interesting that the parabolic system may consist of finitely many maps, but the associated conformal system is infinite. By moving from the finite to the infinite, the analysis becomes easier.

In Chapter 9 we provide a detailed quantative analysis of the dynamical behavior of parabolic maps (in dimension $d \geq 2$) around parabolic points and we apply it to provide a complete characterization of conformal measures of finite parabolic systems in terms of Hausdorff and packing measures. This simultaneously provides the answer to the question about necessary and sufficient conditions for these two geometric measures to be finite and positive.

In first section of the Appendix we collect some basic concepts and theorems from ergodic theory and in the second section contains a compressed exposition of some topics from geometric measure theory which are of interest here.

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We have also provided two indexes, one for terminology and the other for special symbolic notation, and some references. We thank Cambridge University Press for the enormous help provided in bringing this project to fruition. We thank Larry Lindsay for his corrections to parts of the manuscript. Of course, we bear responsibility for all errors and omissions and ask foregiveness of all whom we have overlooked in our credits. Finally, we wish to thank the National Science Foundation for its support for our research during the preparation of this book.

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