1
Preliminaries

Graph directed Markov systems are based upon a directed multigraph and an associated incidence matrix, (V, E, i, t, A). The multigraph consists of a finite set V of vertices and a countable (either finite or infinite) set of directed edges E and two functions $i, t : E \to V$. For each edge e, $i(e)$ is the initial vertex of the edge e and $t(e)$ is the terminal vertex of e. The edge goes from $i(e)$ to $t(e)$. Also, a function $A : E \times E \to \{0, 1\}$ is given, called an incidence matrix. The matrix A is an edge incidence matrix. It determines which edges may follow a given edge. So, the matrix has the property that if $A_{uv} = 1$, then $t(u) = i(v)$. We will consider finite and infinite walks through the vertex set consistent with the incidence matrix. Thus, we define the set of infinite admissible words

$$E_A^\infty = \{\omega \in E^\infty : A_{\omega_i \omega_{i+1}} = 1 \ \text{ for all } i \geq 1\},$$

by E_A^n we denote the set of all subwords of E_A^∞ of length $n \geq 1$, and by E_A^* we denote the set of all finite subwords of E_A^∞. We will drop the subscript A when the matrix is clear from context. We will consider the left shift map $\sigma : E^\infty \to E^\infty$ defined by dropping the first entry of ω. Sometimes we also consider this shift as defined on words of finite length. Given $\omega \in E^*$ by $|\omega|$ we denote the length of the word ω, i.e./ the unique n such that $\omega \in E^n$. If $\omega \in E^\infty$ and $n \geq 1$, then

$$\omega|_n = \omega_1 \dots \omega_n.$$

A *Graph Directed Markov System* (GDMS) consists of a directed multigraph and incidence matrix together with a set of non-empty compact metric spaces $\{X_v\}_{v \in V}$, a number s, $0 < s < 1$, and for every $e \in E$, a 1-to-1 contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with a Lipschitz constant $\leq s$. Briefly, the set

$$S = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e \in E}$$

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is called a GDMS. The main object of interest in this book will be the limit set of the system S and objects associated to this set. We now describe the limit set. For each $\omega \in E_A^*$, say $\omega \in E_A^n$, we consider the map coded by ω:

$$\phi_\omega = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}.$$

For $\omega \in E_A^\infty$, the sets $\{\phi_{\omega|_n}(X_{t(\omega_n)})\}_{n\geq 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n\geq 1} \phi_{\omega|_n}(X_{t(\omega_n)}) \neq \emptyset$. Since for every $n \geq 1$, $\text{diam}\big(\phi_{\omega|_n}(X_{t(\omega_n)})\big) \leq s^n \text{diam}\big(X_{t(\omega_n)}\big) \leq s^n \max\{\text{diam}(X_v) : v \in V\}$, we conclude that the intersection

$$\bigcap_{n\geq 1} \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton and we denote its only element by $\pi(\omega)$. In this way we have defined the *coding map* π:

$$\pi : E_A^\infty \to \bigoplus_{v\in V} X_v$$

from E^∞ to $\bigoplus_{v\in V} X_v$, the disjoint union of the compact sets X_v. The set

$$J = J_S = \pi(E_A^\infty)$$

will be called the *limit set* of the GDMS S. We will also deal with the sets coded by words starting with a given vertex v,

$$J_v = \pi\big(\{\omega \in E^\infty : i(\omega_1) = v\}\big).$$

Obviously $J_S = \bigoplus_{v\in V} J_v$. From now on we will assume that

$$\forall a \in E\, \exists b \in E \quad A_{ab} = 1. \tag{1.1}$$

We extend the functions i, t to E^* by putting

$$i(\omega) = i(\omega_1) \quad \text{and} \quad t(\omega) = t(\omega_{|\omega|}).$$

For each $v \in V$, let $S_v(\infty)$ be the set of limit points of all sequences $\{x_n\}_{n=1}^\infty$, where $x_n \in \phi_{e_n}(X_{t(e_n)})$, for some mutually distinct edges e_n with $i(e_n) = v$. Put

$$S(\infty) = \bigcup_{v\in V} S_v(\infty).$$

We shall prove the following.

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Lemma 1.0.1 *If* $\lim_{e\in E}\text{diam}(\phi_e(X_{t(e)}))=0$*, then for every* $v\in V$

$$\overline{J_v}=J_v\cup S_v(\infty)\cup\bigcup_{\omega\in E_v^*}\phi_\omega(S_{t(\omega)}(\infty)),$$

where $E_v^*=\{\omega\in E^*:i(\omega)=v\}$.

Proof. It follows from the assumption of our lemma and (1.1) that $S_v(\infty)\subset\overline{J_v}$. Thus, if $\omega\in E_v^*$, then

$$\phi_\omega(S_{t(\omega)}(\infty))\subset\phi_\omega(\overline{J_{t(\omega)}})\subset\overline{\phi_\omega(J_{t(\omega)})}\subset\overline{J_v}.$$

Hence, the inclusion

$$J_v\cup S_v(\infty)\cup\bigcup_{\omega\in E_v^*}\phi_\omega(S_{t(\omega)}(\infty))\subset\overline{J_v}$$

is proved. Let $E_v^\infty=\{\omega\in E^\infty:i(\omega_1)=v\}$ and $E_v=\{e\in E:i(e)=v\}$. In order to prove the opposite inclusion to that just given, fix $x\in\overline{J_v}$. Then there exists a sequence $\{\omega^{(n)}\}_{n=1}^\infty$ of points in E_v^∞ such that $x=\lim_{n\to\infty}\pi(\omega^{(n)})$. If the sequence of the first coordinates of the words $\omega^{(n)}$ is infinite, then $x\in S_v(\infty)$ and we are done. So, suppose that the set of the first coordinates is finite. If the set of second coordinates is infinite, then there exists $e_1\in E_v$ and $y\in S_{t(e_1)}(\infty)$ such that $x=\phi_{e_1}(y)$ and we are done in this case too. So, suppose that the set of second coordinates is also finite, but the set of the third coordinates is infinite. Then there exist $e_1\in E_v$, $e_2\in E$ such that $A_{e_1e_2}=1$ and $y\in S_{t(e_2)}(\infty)$ such that $x=\phi_{e_1e_2}(y)$ and we are done. If this procedure halts after finitely many steps, say n, our proof is complete since then $x\in\phi_{e_1e_2\ldots e_n}(S_{t(e_n)}(\infty))$, where $e_1e_2\ldots e_n\in E_v^*$. Otherwise, we will produce a word $\omega\in E_v^\infty$ such that $\text{dist}(x,\phi_{\omega|_n}(X_{t(\omega_n)}))$ tends to zero which implies that $x=\pi(\omega)\in J_v$. □

We end this short introductory chapter by distinguishing two special subclasses of GDMSs. We call a GDMS simply a *graph directed system*(GDS) if $A_{e_1e_2}=1$ if and only if $t(e_1)=i(e_2)$. If, moreover, the set of vertices V is a singleton, then the GDS is called an *iterated function system*. The GDSs with finitely many edges have been introduced in [MW2] (see also [EM]). The infinite (the set of edges is infinite) iterated function systems have been introduced in [MU1] (see also [MU2]).

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2
Symbolic Dynamics

This chapter is of abstract character. By this we mean that we consider a 0-1 incidence matrix $A : I \times I \to \{0, 1\}$ where I is a countable alphabet and

$$E^\infty = \{\omega \in I^\infty : A_{\omega_i \omega_{i+1}} = 1 \quad \text{for all } i \geq 1\}$$

is the space of all A-admissible infinite sequences with terms in I. Thus, in this chapter we are considering a directed graph with vertices the elements of I and a directed edge from i to j if and only if $A_{ij} = 1$. So, in this chapter A determines which vertices may follow a given vertex. The notation E^*, $|\omega|$, $\omega|_n$ and the shift map $\sigma : E^\infty \to E^\infty$ have the same meaning as in Chapter 1. We do not consider graph directed Markov systems here although one may think of the alphabet I as the set of edges of a multigraph. Let us remark that we use the notation E^∞ instead of I_A^∞ in order not to have so many subscripts. Let us fix some more notation and definitions. Given $\omega, \tau \in I^\infty$, we define $\omega \wedge \tau \in I^\infty \cup I^*$ to be the longest initial block common to both ω and τ. For each $\alpha > 0$, we define a *metric* d_α, on I^∞, by setting $d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha|\omega \wedge \tau|}$. These metrics are all equivalent and induce the same topology and Borel sets. A function is uniformly continuous with respect to one of these metrics if and only if it is uniformly continuous with respect to all. Also, a function is Hölder with respect to one of these metrics if and only if it is Hölder with respect to all; of course the Hölder order depends on the metric. If no metric is specifically mentioned, we take it to be d_1.

In this chapter we present various aspects of the thermodynamic formalism of a continuous potential on a shift space generated by a countable alphabet. Our approach has been developed in several papers culminating in [MU3] and stems from that of Ruelle [Ru] and Bowen [B1], cf. also [Wa] and [PU]. The case of a countable shift has also been considered

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(see e.g. [Gu], [GS], [PP], [Sar] and [Zar]). Our definition of topological pressure is more traditional than that proposed in these works. It fits better with our geometric applications and needs no compactifications of the shift space. In particular we are able to construct Gibbs states and equilibrium states of unbounded potentials.

2.1 Topological pressure and variational principles

The incidence matrix A is said to be *irreducible* if for all $i, j \in I$ there exists a path $\omega \in E^*$ such that $\omega_1 = i$ and $\omega_{|\omega|} = j$. This is equivalent to saying that the directed graph is *strongly connected*: for any two elements a, b of I there is a finite path starting at a and ending at b. This is also equivalent to saying that the left shift map, σ, on $E^\infty = I_A^\infty$ is *topologically mixing*: for any two non-empty open subsets U, V of I_A^∞ there is a non-negative integer n such that $\sigma^n(U) \cap V \neq \emptyset$. We say A is *primitive* if there exists $p \geq 1$ such that all the entries of A^p are positive, or in other words, for all $i, j \in I$ there exists a path $\omega \in E^p$ such that $\omega_1 = i$ and $\omega_{|\omega|} = j$. The matrix A is said to be *finitely irreducible* if there exists a finite set $\Lambda \subset E^*$ such that for all $i, j \in I$ there exists a path $\omega \in \Lambda$ for which $i\omega j \in E^*$. We note the following fact. If A is irreducible, then A is finitely irreducible if and only if there is a finite set of letters F such that for every $a \in I$, there are $p, q \in F$ such that $A_{ap} = A_{qa} = 1$. Finally, A is said to be *finitely primitive* if there exists a finite set $\Lambda \subset E^*$ consisting of words of the same length such that for all $i, j \in I$ there exists a path $\omega \in \Lambda$ for which $i\omega j \in E^*$. Notice that a finitely irreducible matrix does not have to be primitive nor conversely. Notice also that the set Λ (associated either with a finitely irreducible or finitely primitive matrix) can be taken to be empty provided E^∞ consists of all infinite words from I. Given a set $F \subset I$, we put

$$E_F^\infty = \{\omega \in E^\infty : \omega_i \in F \text{ for all } i \geq 1\}.$$

A sequence $\{a_n\}_{n=1}^\infty$ consisting of real numbers is said to be subadditive if $a_{n+m} \leq a_n + a_n$ for all $m, n \geq 1$. For the sake of completness we provide the proof of the following well-known elementary fact.

Lemma 2.1.1 *If a sequence $\{a_n\}_{n=1}^\infty$ is subadditive, then $\lim_{n\to\infty} a_n$ exists and is equal to $\inf_n a_n/n$. The limit could be $-\infty$, but if the a_n's are bounded below, then the limit is nonnegative.*

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Proof. Fix $m \geq 1$. Each $n \geq 1$ can be expressed as $n = km + i$ with $0 \leq i < m$. Then

$$\frac{a_n}{n} = \frac{a_{i+km}}{i+km} \leq \frac{a_i}{km} + \frac{a_{km}}{km} \leq \frac{a_i}{km} + \frac{ka_m}{km} = \frac{a_i}{km} + \frac{a_m}{m}$$

If $n \to \infty$ then also $k \to \infty$ and therefore $\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$. Thus $\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf \frac{a_m}{m}$. Now the inequality $\inf \frac{a_m}{m} \leq \liminf_{n\to\infty} \frac{a_n}{n}$ finishes the proof. □

Given a function $f : E_F^\infty \to I\!R$ we define the standard *nth partition function* by

$$Z_n(F,f) = \sum_{\omega\in E^n\cap F^n} \exp\Big(\sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{n-1} f(\sigma^j(\tau))\Big),$$

where $[\omega \cap F] = \{\tau \in E_F^\infty : \tau|_{|\omega|} = \omega\}$. If $F = I$, we simply write $[\omega]$ for $[\omega \cap F]$. We will need the following.

Lemma 2.1.2 *The sequence* $n \mapsto \log Z_n(F,f)$ *is subadditive.*

Proof. We need to show that the sequence $n \mapsto Z_n(F,f)$ is submultiplicative, i.e. that $Z_{m+n}(F,f) \leq Z_m(F,f)Z_n(F,f)$ for all $m,n \geq 1$. And indeed,

$$\begin{aligned}
&Z_{m+n}(F,f)\\
&= \sum_{\omega\in E^{m+n}\cap F^{m+n}} \exp\Big(\sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{mn-1} f(\sigma^j(\tau))\Big)\\
&= \sum_{\omega\in E^{m+n}\cap F^{m+n}} \exp\Big(\sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{m-1} f(\sigma^j(\tau)) + \sum_{j=0}^{n-1} f(\sigma^j(\sigma^m(\tau)))\Big)\\
&\leq \sum_{\omega\in E^{m+n}\cap F^{m+n}} \exp\Big(\sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{m-1} f(\sigma^j(\tau)) + \sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{n-1} f(\sigma^j(\sigma^m(\tau)))\Big)\\
&\leq \sum_{\omega\in E^m\cap F^m} \sum_{\rho\in E^n\cap F^n} \exp\Big(\sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{m-1} f(\sigma^j(\tau)) + \sup_{\gamma\in[\rho\cap F]} \sum_{j=0}^{n-1} f(\sigma^j(\gamma)\Big)\\
&= \sum_{\omega\in E^m\cap F^m} \exp\Big(\sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{m-1} f(\sigma^j(\tau))\Big) \cdot \sum_{\rho\in E^n\cap F^n} \exp\Big(\sup_{\gamma\in[\rho\cap F]} \sum_{j=0}^{m-1} f(\sigma^j(\gamma))\Big)\\
&= Z_m(F,f)Z_n(F,f).
\end{aligned}$$

□

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We can now define the *topological pressure of* f with respect to the shift map $\sigma : E_F^\infty \to E_F^\infty$ to be

$$\mathrm{P}_F(f) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(F,f) = \inf\{\frac{1}{n}\log Z_n(F,f)\}. \tag{2.1}$$

If $F = I$, we suppress the subscript F and write simply $\mathrm{P}(f)$ for $\mathrm{P}_I(f)$ and $Z_n(f)$ for $Z_n(I,f)$.
We let $S_n f$ be the nth partial *orbit sum* of f with respect to σ:

$$S_n f = \sum_{j=0}^{n-1} f \circ \sigma^j.$$

First, we shall provide a characterization of topological pressure expressed in the style of a *Poincaré exponent*.

Theorem 2.1.3 *For every continuous function* $f : E_F^\infty \to I\!R$, *we have*

$$\mathrm{P}_F(f) = \inf\left\{ t \in I\!R : \sum_{\omega\in F^*\cap E^*} \exp\bigl(\sup(S_{|\omega|} f|_{[\omega]})\bigr)\, e^{-t|\omega|} < \infty \right\}.$$

Proof. Fix $t > \mathrm{P}_F(f)$. By the definition of pressure there exists $n_t \geq 1$ such that for every $n \geq n_t$

$$\log \sum_{\omega\in F^n\cap E^n} \exp\bigl(\sup(S_n f|_{[\omega\cap F]})\bigr) < \left(\mathrm{P}_F(f) + \frac{t - \mathrm{P}_F(f)}{2}\right) n$$

and therefore

$$\sum_{\omega\in F^n} \exp\bigl(\sup(S_n f|_{[\omega\cap F]})\bigr) \mathrm{e}^{-tn} \leq \exp\left(\frac{\mathrm{P}_F(f) - t}{2} n\right).$$

Consequently,

$$\sum_{n\geq 0} \sum_{\omega\in F^n\cap E^n} \exp\bigl(\sup(S_n f|_{[\omega\cap F]})\bigr) \mathrm{e}^{-tn} < \infty.$$

Suppose in turn that $t < \mathrm{P}_F(f)$. By the definition of pressure, if n is large enough,

$$\left(\mathrm{P}_F(f) + \frac{t - \mathrm{P}_F(f)}{2}\right) n \leq \log \sum_{\omega\in F^n} \exp\bigl(\sup(S_n f|_{[\omega\cap F]})\bigr)$$

and therefore,

$$\exp\left(\frac{\mathrm{P}_F(f) - t}{2} n\right) \leq \sum_{\omega\in F^n} \exp\bigl(\sup(S_n f|_{[\omega\cap F]})\bigr) \mathrm{e}^{-tn}.$$

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Consequently,

$$\sum_{n\geq 0}\sum_{\omega\in F^n}\exp\bigl(\sup(S_nf|_{[\omega\cap F]})\bigr)\mathrm{e}^{-tn}=\infty.$$

□

There are several things concerning pressure which may differ radically from the case when the alphabet is finite. However, there is a reasonably wide class of functions introduced in [MU3] for which the pressure function is fairly well behaved.

Definition 2.1.4 *(see [MU3]) A function $f: E^\infty \to I\!R$ is acceptable provided it is uniformly continuous and*

$$\mathrm{osc}(f) := \sup_{i\in I}\{\sup(f|_{[i]}) - \inf(f|_{[i]})\} < \infty.$$

Note that an acceptable function need not be bounded. We shall prove the following.

Theorem 2.1.5 *If $f: E^\infty \to I\!R$ is acceptable and A is finitely irreducible, then*

$$\mathrm{P}(f) = \sup\{\mathrm{P}_F(f)\},$$

where the supremum is taken over all finite subsets F of I.

Proof. The inequality $\mathrm{P}(f) \geq \sup\{P_F(f)\}$ is obvious. Let Λ witness that A is finitely irreducible. To prove the converse suppose first that $\mathrm{P}(f) < \infty$. Put $q = \#\Lambda$ and $p = \max\{|\omega| : \omega \in \Lambda\}$ and $T = \min\left\{\inf\sum_{j=0}^{|\omega|-1} f\circ\sigma^j|_{[\omega]} : \omega\in\Lambda\right\}$, where $[\omega] = \{\tau \in E^\infty : \tau|_{|\omega|} = \omega\}$. Fix $\epsilon > 0$. By the acceptability of f, there exists $l \geq 1$ such that $|f(\omega) - f(\tau)| < \epsilon$, if $\omega|_l = \tau|_l$ and $M = \mathrm{osc}(f) < \infty$. Now, fix $k \geq l$. By subadditivity, $\frac{1}{k}\log Z_k(f) \geq \mathrm{P}(f)$. Notice that there exists a finite set $F \subset I$ such that

$$\frac{1}{k}\log Z_k(F,f) > \mathrm{P}(f) - \epsilon. \tag{2.2}$$

We may assume that F contains Λ. Put

$$\overline{f} = \sum_{j=0}^{k-1} f\circ\sigma^j.$$

Now, for every element $\tau = \tau_1,\tau_2,\ldots,\tau_n \in F^k\cap E^k \times\cdots\times F^k\cap E^k$ (n factors) one can choose elements $\alpha_1,\alpha_2,\ldots,\alpha_{n-1}\in\Lambda$ such that $\overline{\tau} = \tau_1\alpha_1\tau_2\alpha_2\ldots\tau_{n-1}\alpha_{n-1}\tau_n \in E^*$. Notice that the function $\tau\mapsto\overline{\tau}$ is at

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most q^{n-1}-to-1 (in fact u^{n-1}-to-1, where u is the number of lengths of words composing Λ. Then for every $n \geq 1$,

$$\begin{aligned}
q^{n-1} & \sum_{i=kn}^{kn+p(n-1)} Z_i(F,f) \\
& \geq \sum_{\tau \in (F^k \cap E^k)^n} \exp\left(\sup_{[\overline{\tau} \cap F]} \sum_{j=0}^{|\overline{\tau}|} f \circ \sigma^j \right) \\
& \geq \sum_{\tau \in (F^k \cap E^k)^n} \exp\left(\inf_{[\overline{\tau}]} \sum_{j=0}^{|\overline{\tau}|} f \circ \sigma^j \right) \\
& \geq \sum_{\tau \in (F^k \cap E^k)^n} \exp\left(\sum_{i=1}^{n} \inf \overline{f}|_{[\tau_i]} + T(n-1) \right) \\
& = \exp(T(n-1)) \sum_{\tau \in (F^k \cap E^k)^n} \exp \sum_{i=1}^{n} \inf \overline{f}|_{[\tau_i]} \\
& \geq \exp(T(n-1)) \sum_{\tau \in (F^k \cap E^k)^n} \exp\left(\sum_{i=1}^{n} (\sup \overline{f}|_{[\tau_i]} - (k-l)\epsilon - Ml) \right) \\
& = \exp(T(n-1) - (k-l)\epsilon n - Mln) \sum_{\tau \in (F^k \cap E^k)^n} \exp \sum_{i=1}^{n} \sup \overline{f}|_{[\tau_i]} \\
& = \mathrm{e}^{-T} \exp\big(n(T - (k-l)\epsilon - Ml)\big) \left(\sum_{\tau \in (F^k \cap E^k)} \exp(\sup \overline{f}|_{[\tau]}) \right)^n .
\end{aligned}$$

Hence, there exists $kn \leq i_n \leq (k+p)n$ such that

$$Z_{i_n}(F,f) \geq \frac{1}{pn} \mathrm{e}^{-T} \exp\big(n(T - (k-l)\epsilon - Ml - \log q)\big) Z_k(F,f)^n$$

and therefore, using (2.2), we obtain

$$\begin{aligned}
\mathrm{P}_F(f) = \lim_{n\to\infty} \frac{1}{i_n} \log Z_{i_n}(F,f) & \geq \frac{-|T|}{k} - \epsilon + \frac{l\epsilon}{k+p} - \frac{Ml + \log p}{k} \\
& + \mathrm{P}(f) - 2\epsilon \geq \mathrm{P}(f) - 7\epsilon
\end{aligned}$$

provided that k is large enough. Thus, letting $\epsilon \searrow 0$, the theorem follows. The case $\mathrm{P}(f) = \infty$ can be treated similarly. $\square$

We say a σ-invariant Borel probability measure $\tilde{\mu}$ on E^∞ is *finitely supported* provided there exists a finite set $F \subset I$ such that $\tilde{\mu}(E_F^\infty) = 1$. The well-known *variational principle* for finitely supported measures (see

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[B1], [Ru], comp. [Wa] and [PU]) tells us that for every finite set $F \subset I$

$$\mathrm{P}_F(f) = \sup\{\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu}\},$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures $\tilde{\mu}$ with $\tilde{\mu}(F^\infty) = 1$ and $\mathrm{h}_{\tilde{\mu}}(\sigma)$ is the *entropy of* $\tilde{\mu}$ with respect to σ. Applying Theorem 2.1.5, we therefore obtain the following.

Theorem 2.1.6 *(1st variational principle) If A is finitely irreducible and if $f : E^\infty \to I\!R$ is acceptable, then*

$$\mathrm{P}(f) = \sup\{(\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu}\},$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures $\tilde{\mu}$ which are finitely supported.

For $n \geq 1$, let α^n be the *standard partition* of E^∞ into cylinders of length n:

$$\alpha^n = \{[\omega] : |\omega| = n\}.$$

If $n = 1$, we write also α for α^1. If β is a countable measurable partition of E^∞ and $\tilde{\mu}$ is a probability measure, then the *entropy of* $\tilde{\mu}$ *with respect to the partition* β is $\mathrm{H}_{\tilde{\mu}}(\beta) = -\sum_{B\in\beta} \tilde{\mu}(B) \log \tilde{\mu}(B)$. Our next theorem is the following.

Theorem 2.1.7 *(2nd variational principle) If $f : E^\infty \to I\!R$ is a continuous function and $\tilde{\mu}$ is a σ-invariant Borel probability measure on E^∞ such that $\int f d\tilde{\mu} > -\infty$, then*

$$\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu} \leq \mathrm{P}(f).$$

In addition, if $\mathrm{P}(f) < \infty$, then there exists $q \geq 1$ such that $\mathrm{H}_{\tilde{\mu}}(\alpha^q) < \infty$.

Proof. If $\mathrm{P}(f) = +\infty$, there is nothing to prove. So, suppose that $\mathrm{P}(f) < \infty$. Then there exists $q \geq 1$ such that $Z_n(f) < \infty$ for every $n \geq q$. Also, for every $n \geq 1$, we have

$$\sum_{|\omega|=n} \tilde{\mu}([\omega]) \sup(S_n f|_{[\omega]}) \geq \int S_n f d\tilde{\mu} = n \int f d\tilde{\mu} > -\infty.$$

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