

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)

Contents

<i>Preface</i>	xiii
Part I Symmetries and Integrals	1
1 Distributions	3
1.1 Distributions and integral manifolds	3
1.1.1 Distributions	3
1.1.2 Morphisms of distributions	4
1.1.3 Integral manifolds	5
1.2 Symmetries of distributions	11
1.3 Characteristic and shuffling symmetries	15
1.4 Curvature of a distribution	18
1.5 Flat distributions and the Frobenius theorem	20
1.6 Complex distributions on real manifolds	23
1.7 The Lie–Bianchi theorem	24
1.7.1 The Maurer–Cartan equations	24
1.7.2 Distributions with a commutative symmetry algebra	27
1.7.3 Lie–Bianchi theorem	30
2 Ordinary differential equations	32
2.1 Symmetries of ODEs	32
2.1.1 Generating functions	32
2.1.2 Lie algebra structure on generating functions	37
2.1.3 Commutative symmetry algebra	38
2.2 Non-linear second-order ODEs	40
2.2.1 Equation $y'' = y' + F(y)$	43
2.2.2 Integration	46
2.2.3 Non-linear third-order equations	48

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)

vi

Contents

2.3	Linear differential equations and linear symmetries	50
2.3.1	The variation of constants method	50
2.3.2	Linear symmetries	51
2.4	Linear symmetries of self-adjoint operators	54
2.5	Schrödinger operators	56
2.5.1	Integrable potentials	58
2.5.2	Spectral problems for KdV potentials	65
2.5.3	Lagrange integrals	73
3	Model differential equations and the Lie superposition principle	76
3.1	Symmetry reduction	76
3.1.1	Reductions by symmetry ideals	76
3.1.2	Reductions by symmetry subalgebras	77
3.2	Model differential equations	78
3.2.1	One-dimensional model equations	80
3.2.2	Riccati equations	82
3.3	Model equations: the series A_k, D_k, C_k	83
3.3.1	Series A_k	83
3.3.2	Series D_k	86
3.3.3	Series C_k	87
3.4	The Lie superposition principle	89
3.4.1	Bianchi equations	92
3.5	\mathcal{AP} -structures and their invariants	94
3.5.1	Decomposition of the de Rham complex	94
3.5.2	Classical almost product structures	96
3.5.3	Almost complex structures	98
3.5.4	\mathcal{AP} -structures on five-dimensional manifolds	98
	Part II Symplectic Algebra	101
4	Linear algebra of symplectic vector spaces	103
4.1	Symplectic vector spaces	103
4.1.1	Bilinear skew-symmetric forms on vector spaces	103
4.1.2	Symplectic structures on vector spaces	104
4.1.3	Canonical bases and coordinates	107
4.2	Symplectic transformations	108
4.2.1	Matrix representation of symplectic transformations	110

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)

<i>Contents</i>		vii
4.3	Lagrangian subspaces	113
4.3.1	Symplectic and Kähler spaces	117
5	Exterior algebra on symplectic vector spaces	119
5.1	Operators \perp and \top	119
5.2	Effective forms and the Hodge–Lepage theorem	125
5.2.1	\mathfrak{sl}_2 -method	132
6	A symplectic classification of exterior 2-forms in dimension 4	135
6.1	Pfaffian	135
6.2	Normal forms	137
6.3	Jacobi planes	142
6.3.1	Classification of Jacobi planes	143
6.3.2	Operators associated with Jacobi planes	145
7	Symplectic classification of exterior 2-forms	147
7.1	Pfaffians and linear operators associated with 2-forms	147
7.2	Symplectic classification of 2-forms with distinct real characteristic numbers	149
7.3	Symplectic classification of 2-forms with distinct complex characteristic numbers	152
7.4	Symplectic classification of 2-forms with multiple characteristic numbers	154
7.5	Symplectic classification of effective 2-forms in dimension 6	160
8	Classification of exterior 3-forms on a six-dimensional symplectic space	162
8.1	A symplectic invariant of effective 3-forms	162
8.1.1	The case of trivial invariants	165
8.1.2	The case of non-trivial invariants	167
8.1.3	Hitchin’s results on the geometry of 3-forms	173
8.2	The stabilizers of orbits and their prolongations	175
8.2.1	Stabilizers	175
8.2.2	Prolongations	178

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)

viii

Contents

Part III Monge–Ampère Equations	181
9 Symplectic manifolds	183
9.1 Symplectic structures	183
9.1.1 The cotangent bundle and the standard symplectic structure	184
9.1.2 Kähler manifolds	186
9.1.3 Orbits and homogeneous symplectic spaces	187
9.2 Vector fields on symplectic manifolds	189
9.2.1 Poisson bracket and Hamiltonian vector fields	189
9.2.2 Canonical coordinates	191
9.3 Submanifolds of symplectic manifolds	192
9.3.1 Presymplectic manifolds	192
9.3.2 Lagrangian submanifolds	194
9.3.3 Involutive submanifolds	197
9.3.4 Lagrangian polarizations	198
10 Contact manifolds	201
10.1 Contact structures	201
10.1.1 Examples	202
10.2 Contact transformations and contact vector fields	208
10.2.1 Examples	209
10.2.2 Contact vector fields	215
10.3 Darboux theorem	219
10.4 A local description of contact transformations	221
10.4.1 Generating functions of Lagrangian submanifolds	221
10.4.2 A description of contact transformations in \mathbb{R}^3	222
11 Monge–Ampère equations	224
11.1 Monge–Ampère operators	224
11.2 Effective differential forms	226
11.3 Calculus on $\Omega^*(C^*)$	230
11.4 The Euler operator	233
11.5 Solutions	236
11.6 Monge–Ampère equations of divergent type	241
12 Symmetries and contact transformations of Monge–Ampère equations	243
12.1 Contact transformations	243

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)

<i>Contents</i>	ix
12.2 Lie equations for contact symmetries	251
12.3 Reduction	256
12.4 Examples	259
12.4.1 The boundary layer equation	259
12.4.2 The thermal conductivity equation	261
12.4.3 The Petrovsky–Kolmogorov–Piskunov equation	262
12.4.4 The Von Karman equation	264
12.5 Symmetries of the reduction	267
12.6 Monge–Ampère equations in symplectic geometry	270
13 Conservation laws	273
13.1 Definition and examples	273
13.2 Calculus for conservation laws	274
13.3 Symmetries and conservations laws	279
13.4 Shock waves and the Hugoniot–Rankine condition	280
13.4.1 Shock Waves for ODEs	280
13.4.2 Discontinuous solutions	281
13.4.3 Shock waves	283
13.5 Calculus of variations and the Monge–Ampère equation	285
13.5.1 The Euler operator	285
13.5.2 Symmetries and conservation laws in variational problems	286
13.5.3 Classical variational problems	287
13.6 Effective cohomology and the Euler operator	288
14 Monge–Ampère equations on two-dimensional manifolds and geometric structures	294
14.1 Non-holonomic geometric structures associated with Monge–Ampère equations	295
14.1.1 Non-holonomic structures on contact manifolds	295
14.1.2 Non-holonomic fields of endomorphisms on generated by Monge–Ampère equations	295
14.1.3 Non-degenerate equations	298
14.1.4 Parabolic equations	302

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)

x

Contents

14.2	Intermediate integrals	304
14.2.1	Classical and non-holonomic intermediate integrals	304
14.2.2	Cauchy problem and non-holonomic intermediate integrals	307
14.3	Symplectic Monge–Ampère equations	308
14.3.1	A field of endomorphisms A_ω on T^*M	308
14.3.2	Non-degenerate symplectic equations	310
14.3.3	Symplectic parabolic equations	312
14.3.4	Intermediate integrals	313
14.4	Cauchy problem for hyperbolic Monge–Ampère equations	313
14.4.1	Constructive methods for integration of Cauchy problem	314
15	Systems of first-order partial differential equations on two-dimensional manifolds	318
15.1	Non-linear differential operators of first order on two-dimensional manifolds	319
15.2	Jacobi equations	321
15.3	Symmetries of Jacobi equations	328
15.4	Geometric structures associated with Jacobi's equations	330
15.5	Conservation laws of Jacobi equations	332
15.6	Cauchy problem for hyperbolic Jacobi equations	334
Part IV	Applications	337
16	Non-linear acoustics	339
16.1	Symmetries and conservation laws of the KZ equation	340
16.1.1	KZ equation and its contact symmetries	340
16.1.2	The structure of the symmetry algebra	342
16.1.3	Classification of one-dimensional subalgebras of $\mathfrak{sl}(2, \mathbb{R})$	345
16.1.4	Classification of symmetries	347
16.1.5	Conservation laws	348
16.2	Singularities of solutions of the KZ equation	349
16.2.1	Caustics	349
16.2.2	Contact shock waves	351

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)*Contents*

xi

17 Non-linear thermal conductivity	356
17.1 Symmetries of the TC equation	356
17.1.1 TC equation	356
17.1.2 Group classification of TC equation	357
17.2 Invariant solutions	363
18 Meteorology applications	371
18.1 Shallow water theory and balanced dynamics	372
18.2 A geometric approach to semi-geostrophic theory	374
18.3 Hyper-Kähler structure and Monge–Ampère operators	376
18.4 Monge–Ampère operators with constant coefficients and plane balanced models	380
Part V Classification of Monge–Ampère equations	383
19 Classification of symplectic MAOs on two-dimensional manifolds	385
19.1 e -Structures	386
19.2 Classification of non-degenerate Monge–Ampère operators	388
19.2.1 Differential invariants of non-degenerate operators	388
19.2.2 Hyperbolic operators	392
19.2.3 Elliptic operators	401
19.3 Classification of degenerate Monge–Ampère operators	406
19.3.1 Non-linear mixed-type operators	406
19.3.2 Linear mixed-type operators	416
20 Classification of symplectic MAEs on two-dimensional manifolds	422
20.1 Monge–Ampère equations with constant coefficients	422
20.1.1 Hyperbolic equations	423
20.1.2 Elliptic equations	425
20.1.3 Parabolic equations	426
20.2 Non-degenerate quasilinear equations	428
20.3 Intermediate integrals and classification	429

Cambridge University Press

978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

Table of Contents

[More information](#)

xii

Contents

20.4	Classification of generic Monge–Ampère equations	430
20.4.1	Monge–Ampère equations and e -structures	430
20.4.2	Normal forms of mixed-type equations	436
20.5	Applications	440
20.5.1	The Born–Infeld equation	440
20.5.2	Gas-dynamic equations	442
20.5.3	Two-dimensional stationary irrotational isentropic flow of a gas	445
21	Contact classification of MAEs on two-dimensional manifolds	447
21.1	Classes $H_{k,l}$	447
21.2	Invariants of non-degenerate Monge–Ampère equations	454
21.2.1	Tensor invariants	454
21.2.2	Absolute and relative invariants	456
21.3	The problem of contact linearization	459
21.4	The problem of equivalence for non-degenerate equations	464
21.4.1	e -Structure for non-degenerate equations	464
21.4.2	Functional invariants	470
22	Symplectic classification of MAEs on three-dimensional manifolds	472
22.1	Jets of submanifolds and differential equations on submanifolds	473
22.2	Prolongations of contact and symplectic manifolds and overdetermined Monge–Ampère equations	476
22.2.1	Prolongations of symplectic manifolds	476
22.2.2	Prolongations of contact manifolds	479
22.3	Differential equations for symplectic equivalence	482
	<i>References</i>	487
	<i>Index</i>	493