Contact Geometry and Nonlinear Differential Equations

Methods from contact and symplectic geometry can be used to solve highly non-trivial non-linear partial and ordinary differential equations without resorting to approximate numerical methods or algebraic computing software. This book explains how it's done. It combines the clarity and accessibility of an advanced textbook with the completeness of an encyclopedia. The basic ideas that Lie and Cartan developed at the end of the nineteenth century to transform solving a differential equation into a problem in geometry or algebra are here reworked in a novel and modern way. Differential equations are considered as a part of contact and symplectic geometry, so that all the machinery of Hodge–de Rham calculus can be applied. In this way a wide class of equations can be tackled, including quasi-linear equations, Monge–Ampère equations (which play an important role in modern theoretical physics and meteorology).

The main features of the book are geometric transparency, clear and almost immediate applications to interesting problems, and exact solutions clarifying how approximate numerical solutions can be better obtained. The types of problem considered range from the classical (e.g., Lie's classification probelm) to the analysis of laser beams or the dynamics of cyclones. The authors balance rigor with the need to solve problems, so it will serve as a reference and as a user's guide.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Volume 101

Contact Geometry and Nonlinear Differential Equations

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Contact Geometry and Non-linear Differential Equations

ALEXEI KUSHNER, VALENTIN LYCHAGIN AND VLADIMIR RUBTSOV

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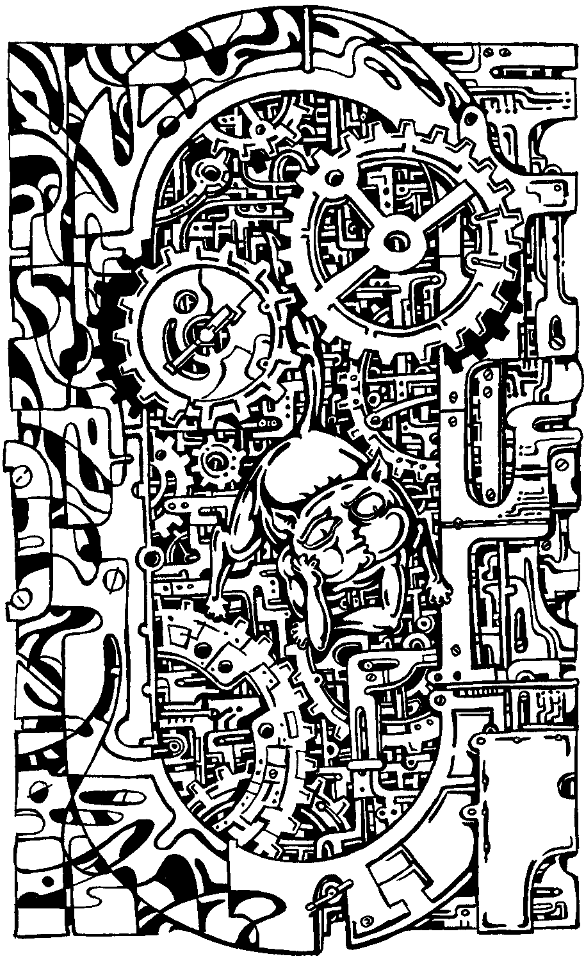
Preface



The aim of this book is to introduce the reader to a geometric study of partial differential equations of second order.

We begin the book with the most classical subject: the geometry of ordinary differential equations, or more general, differential equations of finite type. The main item here is the various notions of symmetry and their use in solving a given differential equation. In Chapter 1 we discuss the distributions, integrability and symmetries. In a form appropriate to our aims, we remind the reader of the main notions of the geometry of distributions: complete integrability, curvature, integral manifolds and symmetries. The Frobenius integrability theorem is presented in its geometric form: as a flatness condition for the distribution.

The main result of this chapter is the famous Lie–Bianchi theorem which gives a condition and an constructive algorithm for integrability in quadratures of a distribution in terms of a Lie algebra of the shuffling symmetries. The theorem clearly explains the etymology of the expression "solvable Lie algebra."

In Chapter 2 we apply these results to explicit integration of scalar ordinary differential equations. We consider some standard examples of differential equations integrable in quadratures but treat them in quite non-standard geometric way to demonstrate the advantage of the language and the method of symmetries. Even in the case of linear differential equations one is able to find some new and interesting results by systematically exploiting the notion of symmetries. The most instructive illustration of this methodology is the application of the linear symmetries of (skew) self-adjoint linear operators. The space of linear symmetries admits in this case the structure of a Lie superalgebra. For example, the even part of the linear symmetries for the Schrödinger operator $L = \partial^2 + W$ is isomorphic to the Lie algebra sl_2, and the generating functions of the linear symmetries satisfy the third-order differential equation. The corresponding third-order operator is the second symmetric power of the Schrödinger one. This operator is also known as a *second Gelfand–Dikii Hamiltonian operator*, which transforms the functional space of the potentials W under appropriate boundary conditions into the infinite-dimensional Poisson algebra known as the *Virasoro algebra*. We use this operator to obtain a description of *integrable* potentials W such that the solutions of the Schrödinger equation $Lu = 0$ can be obtained in quadratures. The operator is also used to find symmetries of the eigenvalue problem for the Schrödinger operator. We show that if the potential W satisfies the KdV equation, or one of their higher analogs, then the eigenvalues and eigenfunctions can be found by quadratures.

In Chapter 3 we illustrate the potency of the geometric approach to the symmetries developing two constructions: a symmetry reduction and Lie's superposition principle.

The first construction is very natural: given an ideal τ of the Lie algebra $\mathfrak{g}$ of shuffling symmetries of a completely integrable distribution, we decompose the integrability problem into two steps: integrability of a completely integrable with symmetry algebra τ, and then the new one with the symmetry factor algebra $\mathfrak{g}/\tau$.

Taking τ to be the radical of $\mathfrak{g}$ we reduce by quadratures the integration of the initial distribution to a distribution with a semi-simple or simple symmetry algebra. The last distributions correspond to ordinary differential equations which we call *model equations*. We give a description of the model equations which correspond to the classical simple Lie algebras. One can see the advantage of using the model system from Lie's *superposition principle*. The principle provides us with *all* the solutions of the model differential equation when we know some finite number of solutions (*a fundamental system of solutions*) and a (in general-non-linear) superposition law. Thus, for the case of the three-dimensional Lie algebra sl_2, the model is the Riccati equation and the superposition rule is given by the *cross-ratio* of four points.

Part II of the book is devoted to *symplectic algebra*. Here we collect necessary information associated with the existence of a symplectic structure on a basic vector space. We had decided to gather together here all the main results not only for the sake of completeness and to make the book self-contained but also because of the conceptual importance of the symplectic structure for Monge–Ampère differential equations.

Historically, the appearance of a symplectic structure in the geometric studies of differential equation has traditionally been attributed to Huygens papers in geometric optics (though, strictly speaking, he had used merely a *contact structure* – an odd-dimensional cousin of the symplectic structure). The importance of symplectic geometry was recognized by J. Lagrange, G. Monge, A. Legendre and especially by Sophus Lie. E. Cartan and his Belgian student T. Lepage had used the symplectic machinery to study the geometry of Monge–Ampère equations at the beginning of the twentieth century. It is curious to note that T. Lepage had introduced a symplectic analog of Hodge theory before the appearance the "very" Hodge decomposition theorem on Riemannian manifolds.

The necessity of symplectic and contact geometry in mechanics is well known. For Monge–Ampère differential equations one should go further and use differential forms in the middle dimension.

The algebra of exterior forms on a symplectic vector spaces has some interesting specific features. In Chapter 5 we study as sl_2- structure given by a couple of "rising" $\top$ and "lowering" $\bot$ operators on the exterior forms:

$$\top\omega = \omega \wedge \Omega, \qquad \bot\,\omega = \imath_{X_\Omega}\omega$$

and by its commutator.

Here Ω is the given symplectic 2-form and $\imath_{X_\Omega}$ is the contraction (the "inner product") with the symplectically dual bivector X_Ω.

Their commutator acts on k–forms by a multiplication:

$$\omega \to (n-k)\omega.$$

The form ω **annihilated** by $\bot$ is called a "primitive" or *effective k-form*. These forms are extremely important – they correspond to Monge–Ampère operators. The arguments of the sl_2-representation theory give the *Hodge – Lepage* decomposition theorem – the main result of this chapter – which states that any exterior k-form ω on the symplectic vector space V is a sum of the forms $\omega_i \wedge \Omega^i, i = 0, \ldots,$ where ω_i are effective forms uniquely determined by ω.

The classification problems for differential equations and operators have their trace in linear algebra – this is a classification of effective forms with respect to the symplectic group. Chapter 6 deals with the easiest classifications problems in dimension 4. In the next chapter we give a symplectic classification of exterior 2-forms in arbitrary dimensions.

In Chapter 8 we classify effective 3-forms in six-dimensional symplectic space with respect to a natural action of the symplectic group Sp_3 . The problem has a long history. Being in the spirit of the classical questions of the theory of geometric invariants, this problem was well known within a classification of spinors of dimension 12 and 14 (see [40], [90]) for the case when the base field is algebraically closed. Their methods do not work for the *real* classification. The first classification was obtained in our papers [74, 77] where the list of normal forms had a gap that was later filled by B. Banos [4, 5].

Part IV is devoted to the Monge–Ampère equations and to the related objects: Monge–Ampère operators and partial differential equation (PDE) systems on two-dimensional manifolds.

Chapters 9 and 10 contain some necessary information about symplectic and contact manifolds.

The application of the algebraic machinery of Chapter 5 gives a description of the Monge–Ampère equations and Monge – Ampère operators. The initial point of our approach is the following observation: to any differential k-form $\omega \in \Omega^k(J^1M)$, where J^1M is the space of 1-jet functions on a manifold M, we attach a non-linear second-order differential operator $\Delta_\omega : C^\infty(M) \to \Omega^k(M)$, acting as

$$\Delta_\omega(h) = j_1(h)^*(\omega),$$

where $j_1(h) : M \to J^1M$ is the 1-jet prolongation of a function $h \in C^\infty(M)$.

We see that the first advantage of this approach is a reduction of the order of the jet spaces: we use a simpler space J^1M instead of the space J^2M where the Monge–Ampère equations should lie *ad hoc*, being second-order partial differential equations. The space J^1M has the Cartan distribution which in this case is nothing but the aforementioned contact structure which impacts fascinatingly on the treatment of second-order differential operators and equations.

We should stress that our definition does not cover *all* non-linear second-order differential equations but only a certain subclass of them. This subclass is rather wide and contains all linear, quasi-linear and Monge–Ampère equations. We call the operators Δ_ω with $\omega \in \Omega^n(M)$, where $n = \dim M$, *Monge–Ampère operators*. The following observation justifies this definition: being written in a local canonical contact coordinates on J^1M the operators Δ_ω have the same type of non-linearity as the Monge – Ampère operators. Namely, the non-linearity involves the determinant of the Hesse matrix and its minors.

The correspondence $\omega \to \Delta_\omega$ is not one-to-one: this map has a huge kernel. If we denote the canonical contact 1-form on J^1M by ω_0, then the kernel is generated by the forms $\alpha \wedge \omega_0 + \beta \wedge d\omega_0$.

It is not hard to check that these forms produce an ideal $\mathcal{C}$ in the exterior algebra $\Omega^*(J^1M)$ which we call Cartan ideal, and the quotient $\Omega^*(J^1M)/\mathcal{C}$ by this ideal is isomorphic to the *effective* exterior forms $\Omega^*_\epsilon(J^1M)$ which we had discussed above. Hence, the effective exterior forms uniquely define Monge – Ampère operators and we can apply all of the machinery of contact/symplectic geometry to a study of these operators and the related non-linear differential equations. For example, from the geometrical point of view, solutions of differential equations corresponding to Δ_ω are nothing but the Legendre submanifolds L in J^1M which are *integral* with respect to the form ω, that is, $\omega|_L = 0$. It is also much easier to apply the contact transformations to differential forms than to the differential operators, so one can define (infinitesimal) symmetries of the Monge–Ampère operators and Monge–Ampère equations by using the induced action of the contact diffeomorphisms (respectively contact vector fields) on the effective differential forms.

In Chapter 11 we introduce and discuss some operators acting on the effective forms and (by correspondence) on the Monge – Ampère operators. First of all the de Rham operator induces a complex on the algebra of effective forms. The cohomology of the complex coincides with the de Rham cohomology of the base M up to dimension $n-1$, where $n = \dim M$. They are trivial in dimensions greater than n, and only in dimension n do we have an essential difference with the cohomology of M. This relates to conservation laws and an Euler operator $\mathcal{E}$. By a conservation law we mean an $(n-1)$-differential form θ on J^1M, such that $d\theta|_L = 0$ for all solutions L. One can check that this is possible if and only if $d\theta = g\omega \bmod \mathcal{C}$ for some function $g \in C^\infty(J^1M)$. We call such a function a *generating function* of the conservation law. There is one-to-one correspondence between generating functions and conservation laws considered up to the trivial ones, and a function g is a generating function if and only if $\mathcal{E}(g\omega) = 0$. We study conservation laws in Chapter 13. We show a relation between contact symmetries and conservation laws for Monge–Ampère equations of divergent type that generalize the classical Noeter theorem in variational calculus. Conservation laws can be used in different directions and here we discuss their application to the classical problem of "sewing" of two solutions by a border of codimension 1. This leads us to the contact analog of the classical Hugoniot–Rankin conditions which are used further for description of shock waves and discontinuous solutions. The end of this chapter is devoted to an application of the developed approach to variational problems. We show that the Euler operator is exactly the operator in the corresponding Euler–Lagrange equations. The chapter closes with a description of non-holonomic filtration in the exterior algebra of J^1M and with an interpretation of the Euler operator as a connecting differential in the spectral sequence associated with this filtration.

Chapter 14 deals with the special case when the base manifold M is two-dimensional. This situation has a lot of complementary geometric and algebraic structures which come into play. The structures resulte from the naturally defined non-holonomic field of endomorphism on J^1M, that is, a field of operators defined on the Cartan distribution only. This leads to a *non-holonomic almost complex structure,* for elliptic differential equations, and to a *non-holonomic almost product structure* for the case of hyperbolic equations. For the parabolic one we get a *non-holonomic almost tangent structure*.

The theory becomes more enlightened in the case when J^1M is replaced by the cotangent bundle T^*M. We call the corresponding Monge–Ampère equation *symplectic* and their geometry is defined by the corresponding structure on the phase space. Thus, for example, elliptic equations define an almost complex structure on T^*M. The liaison between the geometric structures and equations will be used profoundly in subsequent chapters to establish and to clarify the classification and equivalence problems.

Part IV of the book has a somewhat specific feature – on one hand it is so important and voluminous that it could be chosen as the foundation for a separate book. On the other hand, we had decided to include this part as an illustration how the proposed approach can be effectively used in practice for many different type of equations, coming from practically all branches of the natural sciences. We were clearly unable to cover all possible applications but we had focused on the examples which were elaborated by ourselves during long-time contacts with physicists, engineers, biologists, etc. Most of the examples were developed up to numerical results and had figured as a useful part of some joint technical and scientific projects which were undertaken at the Applied Mathematical Department of Moscow Technical University of Civil Constructions during the period 1978–1990 under the direction of one of the authors (VL) and with a strong participation of two others. Some of the examples in this part were developed at Astrakhan State University under the guidance of one of the authors (AK) in a collaboration with the Biology Department of Moscow State University between 1999 and 2003. Another important collaboration which we acknowledge in this part is the long-time cooperation of third author (VR) with applied mathematicians and meteorologists from the Meteorological Office (Bracknell, UK) and Reading University.

Chapter 15 is devoted to a study of the Khokhlov–Zabolotskaya (KZ) equation. We knew about this equation in the mid-1970s from contacts with the theoretical acoustics group of R. Khokhlov (Physics Department of Moscow State University). The equation describes the propagation of three-dimensional sound beams in a non-linear medium. We treat this equation in its full three-dimensional version. It worth mentioning that the two-dimensional version of this equation is a hierarchy member of the famous dispersionless Kadomtsev–Petviashvili integrable system. The numerous applications and physically relevant versions of this dispersionless hierarchy are beyond the scope of our book (for statements and references see [78]).

We describe symmetries, conservation laws and exact solutions of the KZ equation. We discuss singularities of the solutions, Hugoniot–Rankin conditions and shock acoustic waves. Using this information we give a mathematical explanation to an experimentally verified phenomena of self-diffraction and periodic oscillation of sound beams (which is completely similar to the behavior of beams) and give some explicit formulas for the parameters of this behavior.

A version of the Kolmogorov–Petrovsky–Piskunov equation with a non-linear diffusion coefficient is the subject of our study in Chapter 16. This equation has a lot of interest in biology, ecology, and heat and mass transfer theory. We compute the Lie algebra of its symmetries and show how to use them to construct invariant solutions.

In Chapter 17 we collect all the applications of geometric studies of Monge–Ampère equations in theoretical meteorology where Monge–Ampère like operators had appeared some time ago in so-called *semi-geostrophic* models. They constitute an important class of models which are very useful in numerical weather prediction. This chapter is based on some on-going research by one of the authors with I. Roulstone (Surrey University, UK). We give a short account of the geometric study of balanced rotational models, which mathematically means a very special case of the Navier–Stokes system with the presence of Coriolis-like forces. An important aero- and hydrodynamical notion associated with these models is a *potential vorticity*. The conservation law of this quantity (under some mild restrictions) gives a non-linear differential equation which is easily represented in a form of the symplectic Monge–Ampère equation. The geometric structures related to this symplectic Monge–Ampère equation define and sometimes (and in turn are defined by) some nearly balanced two-dimensional model.

The second main motive of this book is a contact equivalence problem for differential equations. The first and important algebraic step to this problem was made in the previous parts of the book when we had discussed the classification and equivalence for (effective) exterior forms.

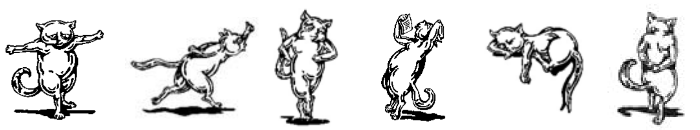
Part V of the book contains the contact classification results on Monge–Ampère equations (in analytic and smooth categories) which can be obtained by our geometric approach and which have the most complete form when the base manifolds are two- or three-dimensional.

The first case ($\dim M = 2$) can be attributed as a *classical Sophus Lie problem*. This problem was raised in the S. Lie article [66] and in our language it may be reformulated in the following way: *to find the equivalence classes of second-order (non-linear) differential equations with respect to the (local) group of contact diffeomorphisms.* Lie himself had stated some theorems (or without proofs or with some indications/hints of them) which can be considered as an attempt to give answers to the problem in some special cases. One of his main results is a statement about (quasi-)linearizability of any *analytic* Monge–Ampère equation. He had also considered the Monge–Ampère equation with constant coefficients and the Monge–Ampère equation in the presence of so-called *intermediate integrals*.

The essential inroad to the classical Lie problem was given by the French School and mainly by G. Darboux and E. Goursat who gave a classification of the two-dimensional hyperbolic Monge–Ampère equation under some restrictions. We should mention that the geometric version of Goursat's results was given by T. Morimoto who had used the language of G-structures and whose approach differs cardinally from ours (see [83]).

Chapters 18–21 contain a modern version of S. Lie's results (which to our knowledge were never completely proven until our papers in the early 1980s) and more general results on the symplectic and contact classification (A. Kushner, B. Kruglikov and D. Tunitskii), based on e-structures naturally defined by Monge–Ampère equations of the general type. Concerning the general equivalence problem we outline classification results for Monge–Ampère equations in the general case. We apply these results for classification and normal forms of symplectic Monge–Ampère equations on three-dimensional manifolds.

You will discover a family of sympathic cats which decorate the main text. Each cat has his own personal name and we hope that it will not be an enigmatic problem to our cleverminded and brilliant readers to understand why one or another cat appears in its proper place place in the text. A list of pictures of these cats with their names appear below.



Welcome Eureka Mentor Thinking Lazy Terminator

In conclusion we wish to thank our friends, colleagues and students for their help and support during the preparation of this book. In particular, we wish to thank Marat Djamaletdinov for his beautiful pictures which, we believe, should captivate readers.