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PART I

Symmetries and Integrals

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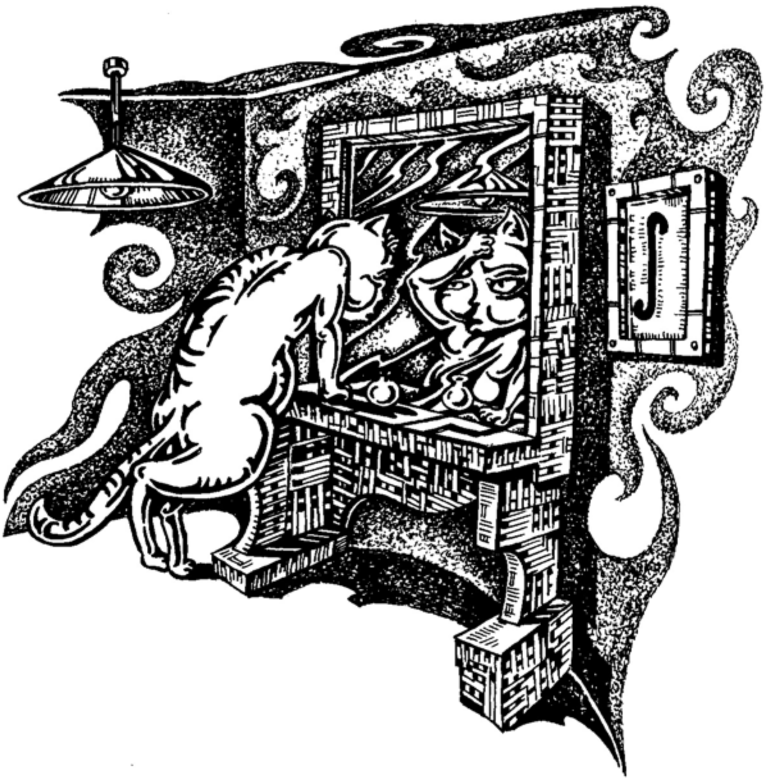

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1
Distributions

1.1 Distributions and integral manifolds

1.1.1 Distributions

Let M be an $(n+m)$-dimensional smooth manifold, and let $\tau : TM \to M$ be the tangent bundle. By a *distribution* P on M one means a *smooth* field $P : a \in M \longmapsto P_a = P(a) \subset T_aM$ of m-dimensional subspaces of the tangent spaces. The number m is called a *dimension* of the distribution, $m = \dim P$, and n is called a *codimension* of P, $n = \operatorname{codim} P$.

There are different ways to say that P is a smooth field, and to define a distribution. We give the more important ones.

1. **As subbundles**
   Let $E_P = \bigcup_{a \in M} P(a) \subset TM$. Then P is a distribution if and only if $\tau_P : E_P \to M$ is a subbundle of the tangent bundle.
2. **By local bases**
   We say that a vector field $X \in D(M)$ *belongs* (or is *tangent*) to P on a subset $N \subset M$, if $X_a \in P(a)$ for all $a \in N$. Then smoothness of P means that there are local bases for P consisting of vector fields that belong to P.

   In other words, for any point $a \in M$ there exists a neighborhood $\mathcal{O}$ of a and m vector fields $X_1, \ldots, X_m$ that belong to P on $\mathcal{O}$ and such that vectors $X_{1,b}, \ldots, X_{m,b}$ form a basis of P_b at any $b \in \mathcal{O}$. Note, that the condition *X belongs* to P means that the vector field X is a section of the bundle $\tau_P : E_P \to M$.

   We denote by $D(P)$ the set of all vector fields that belong to P. It is clear that $D(P)$ is a module over the algebra $C^\infty(M)$ of smooth functions on M:

$$X, Y \in D(P) \Longrightarrow X + Y \in D(P),$$
$$f \in C^\infty(M), \quad X \in D(P) \Longrightarrow fX \in D(P).$$

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In the case when $D(P)$ admits a global basis, consisting, say, of vector fields $X_1, \dots, X_m$ we write $P = \mathcal{F}\langle X_1, \dots, X_m \rangle$.

3. **By equations**
Let $\mathrm{Ann}(P(a)) \subset T_a^*M$ be the *annihilator* of $P(a) \subset T_aM$, that is

$$\mathrm{Ann}(P(a)) = \{\omega_a \in T_a^*M | \omega_a = 0 \text{ on } P(a)\}.$$

In other words, $\mathrm{Ann}(P(a))$ contains all linear equations for $P(a)$, that is all covectors ω_a vanishing on vectors from $P(a)$. Note that $\dim \mathrm{Ann}(P(a)) = n$.

We say that a differential 1-form $\omega \in \Omega^1(M)$ *annihilates* P on a subset $N \subset M$ if and only if $\omega_a \in \mathrm{Ann}(P(a))$ for all $a \in N$. We denote by $\mathrm{Ann}(P)$ the set of all differential forms on M that annihilate P and by $\Omega^1(M)$ the $C^\infty(M)$-module of differential 1-forms on M. Obviously, $\mathrm{Ann}(P)$ is a module over $C^\infty(M)$:

$$\alpha, \beta \in \mathrm{Ann}(P) \Longrightarrow \alpha + \beta \in \mathrm{Ann}(P),$$
$$f \in C^\infty(M), \quad \alpha \in \mathrm{Ann}(P) \Longrightarrow f\alpha \in \mathrm{Ann}(P).$$

In this terms the smoothness of P means that locally P can be defined by n differential 1-forms; that is, for any point $a \in M$ there exists a neighborhood $\mathcal{O}$ of a and n differential 1-forms $\omega_1, \dots, \omega_n$ that annihilate P on $\mathcal{O}$ and such that covectors $\omega_{1,b}, \dots, \omega_{n,b}$ form a basis of $\mathrm{Ann}(P(b))$ at any $b \in \mathcal{O}$. In the case when $\mathrm{Ann}(P)$ admits a global basis, say, $\omega_1, \dots, \omega_n$ we write

$$P = \mathcal{F}\langle \omega_1, \dots, \omega_n \rangle.$$

For the distribution $P = \mathcal{F}\langle X_1, \dots, X_m \rangle$ one can define its derivatives. The distribution $P^{(1)}$ on M which is generated by the vector fields $X_1, \dots, X_m$ and by all possible sorts of commutators $[X_i, X_j]$ $(i < j; i, j = 1, \dots, m)$ is called the *first derivative* of P, i.e.,

$$P^{(1)} = \mathcal{F}\langle X_1, \dots, X_m, [X_1, X_2], \dots, [X_1, X_m], \dots, [X_{m-1}, X_m] \rangle.$$

Analogously one can define the higher derivatives: $P^{(k+1)} \stackrel{\text{def}}{=} \left(P^{(k)}\right)^{(1)}$ $(k > 1)$.

1.1.2 Morphisms of distributions

Let $F : N \to M$ be a smooth map and let P be a distribution on M. Then differentials

$$F_{*,b} : T_bN \to T_aM$$

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where $a = F(b)$, define a family $F^*(P)$ of vector spaces on N by

$$F^*(P)(b) = F_*^{-1}(P(a))$$

for all $b \in N$.

Dimension $\dim F^*(P)(b)$ is equal to $\dim\ker F_{*,b} + \dim(\operatorname{Im} F_{*,b} \cap P(a))$ and therefore varies in the general case.

Note that if $\omega_{1,a}, \ldots, \omega_{n,a}$ is a basis in $\operatorname{Ann}(P(a))$ then $F^*(\omega_{1,a}), \ldots, F^*(\omega_{n,a})$ generates $\operatorname{Ann}(F^*(P)(b))$.

We say that a map F is *P-regular* if the dimension function $b \in N \longmapsto \dim F^*(P)(b)$ is locally constant.

For the case of P-regular maps $F^*(P)$ is a distribution on N. We call this distribution the *image of P under F*.

The following three cases have great importance in applications.

1. F is a diffeomorphism, then the image $F^*(P)$ is well defined for any distribution P.
2. F is a surjection, or a smooth bundle. Then the image $F^*(P)$ is well defined for any distribution P also.
3. F is an embedding. Thus N is a submanifold of M. Then regularity F means that the intersection $T_bN \cap P(b)$ has a constant dimension for all $b \in N$. In this case we call $F^*(P)$ the *restriction of P* on N.

1.1.3 Integral manifolds

Let P be a distribution on M. A submanifold $i : N \hookrightarrow M$ is said to be *integral* for P if the restriction of P to N is equal to the tangent bundle, that is,

$$T_aN \subset P(a) \tag{1.1}$$

for any point $a \in N$.

This definition implies that the dimension of an integral manifold cannot exceed the dimension of a distribution.

If we define distributions in terms of differential 1-forms, say, locally $P = \mathcal{F}\langle\omega_1, \ldots, \omega_n\rangle$ then condition (1.1) takes the form

$$\omega_1|_N = 0, \ldots, \omega_n|_N = 0$$

of the so-called *Pfaff system*.

An integral manifold N is called a *maximal integral manifold* if for any point $a \in N$ one can find a neighborhood $\mathcal{O}$ of a such that there is no integral manifold N' such that $N' \supset N \cap \mathcal{O}$.

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From this it is clear that the dimension of maximal integral manifolds does not exceed the dimension of the distribution. The distributions which have integral manifolds of the dimension equal to the dimension of the distribution are the simpler ones. They are called *completely integrable distributions*, or *CIDs*.

A smooth function $H \in C^\infty(M)$ is called *first integral* for P if $dH \in \operatorname{Ann}(P)$.

As we will see later on a distribution P, is completely integrable if and only if it has (locally) $n = \operatorname{codim} P$ first functional independent integrals $H_1, \ldots, H_n$; that is, (locally) $P = \mathcal{F}\langle dH_1, \ldots, dH_n\rangle$.

Note that any distribution has at least one-dimensional integral manifolds (integral curves). Indeed, any integral curve of a vector field $X \in D(P)$ is an integral curve of the distribution P. To find this curve one should solve some ordinary differential equations (ODEs). This observation has a general nature; namely, the problem of finding integral manifolds of a distribution implies a solution of some differential equation, and vice versa, the problem of finding solutions of differential equations is equivalent to finding integral manifolds of some distributions.

Let us look at a few examples.

Example 1.1.1 *The simplest non-trivial distribution is a one-dimensional distribution on the plane $M = \mathbb{R}^2$. Let x, y be coordinates on M, and let $P = \mathcal{F}\langle\omega\rangle$ where $\omega = a(x, y)\, dx + b(x, y)\, dy$, and $a^2 + b^2 \neq 0$. Let $N \subset M$ be an integral curve. Assume, for example, that x can be chosen as a (local) coordinate on N. Then N is a graph of a function $h(x)$,*

$$N = \{(x, h(x)), x \in \mathbb{R}\}$$

and the Pfaff equation $\omega|_N = 0$ takes the form of the first-order differential equation

$$a(x, h(x)) + b(x, y(x))y'(x) = 0.$$

Note also that $P = \mathcal{F}\langle X\rangle$ where

$$X = b(x, y)\frac{\partial}{\partial x} - a(x, y)\frac{\partial}{\partial y}$$

and therefore to find integral curves of the distribution one should solve the system of differential equations:

$$\dot{x} = b(x, y), \quad \dot{y} = -a(x, y).$$

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The existence theorem shows that these equations have smooth solutions and therefore the distribution has integral manifolds of dimension 1*. Thus* P *is a CID.*

We observe from this example that integration of a first-order ODE, say,

$$y' = F(x, y)$$

is equivalent to finding integral curves of the distribution $P = \mathcal{F}\langle\omega\rangle$ where

$$\omega = dy - F(x, y)dx.$$

In this case we also have

$$P = \mathcal{F}\langle D\rangle$$

where

$$D = \frac{\partial}{\partial x} + F(x, y)\frac{\partial}{\partial y}.$$

The next example generalizes this observation for ODEs of arbitrary order.

Example 1.1.2 (Cartan distribution) *Let* $M = \mathbb{R}^{k+1}$*. Denote the coordinates in* M *by* $x, p_0, p_1, \ldots, p_k$ *and given a function* $F(x, p_0, \ldots, p_{k-1})$ *consider the following differential 1-forms:*

$$\begin{aligned}
\omega_0 &= dp_0 - p_1\, dx,\\
\omega_1 &= dp_1 - p_2\, dx,\\
\vdots \quad & \quad \vdots \quad\quad \vdots \quad\quad \vdots\\
\omega_{k-2} &= dp_{k-2} - p_{k-1}\, dx,\\
\omega_{k-1} &= dp_{k-1} - F(x, p_0, \ldots, p_{k-1})\, dx
\end{aligned}$$

and the distribution $P = \mathcal{F}\langle\omega_0, \ldots, \omega_{k-1}\rangle$*. This is the one-dimensional distribution, called the Cartan distribution.*

This distribution can also be described by a single vector field D*,* $P = \mathcal{F}\langle D\rangle$*, where*

$$D = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial p_0} + p_2\frac{\partial}{\partial p_1} + \cdots + p_{k-1}\frac{\partial}{\partial p_{k-2}} + F(x, p_0, \ldots, p_{k-1})\frac{\partial}{\partial p_{k-1}}.$$

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If N is an integral curve of the distribution then x can be chosen as a coordinate on N, and therefore

$$N = \{(x, h_0(x), h_1(x), \ldots, h_{k-1}(x)),\ x \in \mathbb{R}\}.$$

Conditions

$$\omega_0|_N = 0, \quad \ldots, \quad \omega_{k-2}|_N = 0$$

imply that

$$h_1 = h_0', \quad h_2 = h_1', \quad \ldots, \quad h_{k-1} = h_{k-2}'$$

or that

$$N = \left\{\left(x, h(x), h'(x), \ldots, h^{(k-1)}(x)\right), x \in \mathbb{R}\right\}$$

for some function $h(x)$.

The last equation $\omega_{k-1}|_N = 0$ gives us an ordinary differential equation

$$h^{(k)}(x) = F\left(x, h(x), h'(x), \ldots, h^{(k-1)}(x)\right).$$

The existence theorem shows us once more that the integral curves do exist, and therefore the Cartan distribution is a CID.

Note that three is the lowest number of dimensions where one can encounter a non-CID.

Example 1.1.3 (Contact distribution, see Figure 1.1) *Let $M = \mathbb{R}^{2n+1}$ and let $P = \mathcal{F}\langle\omega\rangle$ where*

$$\omega = du - \sum_{i=1}^{n} p_i\, dq_i$$

in the coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ on $\mathbb{R}^{2n+1}$.

Then P is a $2n$-dimensional distribution, but there are no $2n$-dimensional integral manifolds of P. Let us assume, for example, that $n = 1$, and that N is a two-dimensional integral manifold such that q and p, say, are (local) coordinates on N. Then N given by $u = h(q, p)$ for some function $h(q, p)$ and $\omega|_N = 0$ imply contradictory differential equations for h:

$$h_q = p, \quad h_p = 0.$$

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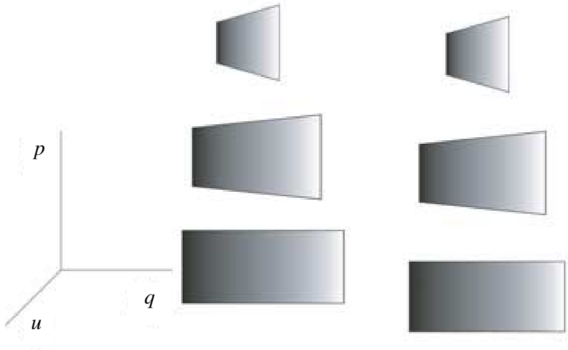

Figure 1.1. The contact distribution in $\mathbb{R}^3$.

On the other hand, every smooth function $f(q_1,\ldots,q_n)$ *determines an n-dimensional submanifold*

$$L_f = \left\{ u = f(q), p_1 = \frac{\partial f}{\partial q_1}, \ldots, p_n = \frac{\partial f}{\partial q_n} \right\}$$

which is integral because of

$$\omega|_{L_f} = 0.$$

Note that this distribution can also be defined by $2n$ *vector fields*

$$X_1 = \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial u}, \quad \ldots, \quad X_n = \frac{\partial}{\partial q_n} + p_n \frac{\partial}{\partial u},$$

$$Y_1 = \frac{\partial}{\partial p_1}, \quad \ldots, \quad Y_n = \frac{\partial}{\partial p_n}.$$

This example introduces the special case of a *contact distribution*, which is of great importance throughout this book.

Example 1.1.4 (Oricycle distribution) *Let* $M = \mathbb{R} \times \mathbb{R}^+ \times S^1$ *be a three-dimensional manifold. The following differential form*

$$\omega = (1 - \cos\phi)\, dx + \sin\phi\, dy - y\, d\phi$$

determines the so-called oricycle distribution on M*.*

Here $x, y > 0$ *are coordinates on* $\mathbb{R} \times \mathbb{R}^+$ *and* ϕ *is the angle on* S^1*.*

Integral curves of this distribution are oricycles.

Example 1.1.5 *Let* M *be a Möbius strip (see Figure 1.2). Define a one-dimensional distribution* P *on* M *where* $P(a)$ *is the line perpendicular to the*

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Figure 1.2. The distribution on the Möbius strip.

central circle of the strip. Both modules $D(P)$ and $\mathrm{Ann}(P)$ associated with the distribution are not free because the fibering of the Möbius strip over its central circle is non-trivial. But both modules become free as soon as we cut the bundle in such a way that it becomes trivial. Integral curves of this distribution are the fibres of the bundle $M \to S^1$.

Example 1.1.6 (Overdetermined system of PDEs) *Consider the following overdetermined system of partial differential equations (PDEs):*

$$\begin{aligned}\phi_x &= A(x, y, \phi, \psi),\\ \phi_y &= B(x, y, \phi, \psi),\\ \psi_x &= C(x, y, \phi, \psi),\\ \psi_y &= D(x, y, \phi, \psi),\end{aligned}$$

with respect to functions $\phi(x, y)$ and $\psi(x, y)$.

Define two differential 1-forms

$$\omega_1 = du - A(x, y, u, v)\, dx - B(x, y, u, v)\, dy$$

and

$$\omega_2 = dv - C(x, y, u, v)\, dx - D(x, y, u, v)\, dy$$

on the space $\mathbb{R}^4$ with coordinates x, y, u, v.

Then the pair of functions $(\phi(x, y), \psi(x, y))$ is a solution of the system if and only if the surface

$$\Gamma = \{u = \phi(x, y),\ v = \psi(x, y)\} \subset \mathbb{R}^4$$

is an integral manifold of the distribution $\mathcal{F}\langle\omega_1, \omega_2\rangle$.

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