I. G. Macdonald
Queen Mary, University of London

Affine Hecke Algebras and Orthogonal Polynomials

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13 pt *System* LaTeX 2ε [TB]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

Macdonald, I. G. (Ian Grant)
Affine Hecke algebras and orthogonal polynomials / I.G. Macdonald.
p. cm. – (Cambridge tracts in mathematics; 157)
Includes bibliographical references and index.
ISBN 0 521 82472 9 (hardback)
1. Hecke algebras. 2. Orthogonal polynomials. I. Title. II. Series.
QA174.2 .M28 2003
512′.55 – dc21 2002031075

ISBN 0 521 82472 9 hardback

Contents

1

Affine root systems

1.1 Notation and terminology

Let E be an affine space over a field K: that is to say, E is a set on which a K-vector space V acts faithfully and transitively. The elements of V are called *translations* of E, and the effect of a translation $v \in V$ on $x \in E$ is written $x + v$. If $y = x + v$ we write $v = y - x$.

Let E' be another affine space over K, and let V' be its vector space of translations. A mapping $f : E \to E'$ is said to be *affine-linear* if there exists a K-linear mapping $Df : V \to V'$, called the *derivative* of f, such that

$$f(x + v) = f(x) + (Df)(v). \tag{1.1.1}$$

for all $x \in E$ and $v \in V$. In particular, a function $f: E \to K$ is affine-linear if and only if there exists a linear form $Df: V \to K$ such that (1.1.1) holds.

If $f, g : E \to K$ are affine-linear and $\lambda, \mu \in K$, the function $h = \lambda f + \mu g : x \mapsto \lambda f(x) + \mu g(x)$ is affine-linear, with derivative $Dh = \lambda Df + \mu Dg$. Hence the set F of all affine-linear functions $f: E \to K$ is a K-vector space, and D is a K-linear mapping of F onto the dual V^* of the vector space V. The kernel of D is the 1-dimensional subspace F^0 of F consisting of the constant functions.

Let F^* be the dual of the vector space F. For each $x \in E$, the evaluation map $\varepsilon_x : f \mapsto f(x)$ belongs to F^*, and the mapping $x \mapsto \varepsilon_x$ embeds E in F^* as an affine hyperplane. Likewise, for each $v \in V$ let $\varepsilon_v \in F^*$ be the mapping $f \mapsto (Df)(v)$. If $v = y - x$, where $x, y \in E$, we have $\varepsilon_v = \varepsilon_y - \varepsilon_x$ by (1.1.1), and the mapping $v \mapsto \varepsilon_v$ embeds V in F^* as the hyperplane through the origin parallel to E.

From now on, K will be the field $\mathbb{R}$ of real numbers, and V will be a real vector space of finite dimension $n > 0$, equipped with a positive definite symmetric scalar product $<u, v>$. We shall write

$$|v| = <v, v>^{1/2}$$

for the length of a vector $v \in V$. Then E is a Euclidean space of dimension n, and is a metric space for the distance function $d(x, y) = |x - y|$.

We shall identify V with its dual space V^* by means of the scalar product $<u, v>$. For any affine-linear function $f\colon E \to \mathbb{R}$, (1.1.1) now takes the form

$$f(x + v) = f(x) + <Df, v> \tag{1.1.2}$$

and Df is the *gradient* of f, in the usual sense of calculus.

We define a scalar product on the space F as follows:

$$<f, g> = <Df, Dg>. \tag{1.1.3}$$

This scalar product is positive semidefinite, with radical the one-dimensional space F^0 of constant functions.

For each $v \neq 0$ in V let

$$v^\vee = 2v/|v|^2$$

and for each non-constant $f \in F$ let

$$f^\vee = 2f/|f|^2.$$

Also let

$$H_f = f^{-1}(0)$$

which is an affine hyperplane in E. The reflection in this hyperplane is the isometry $s_f\colon E \to E$ given by the formula

$$s_f(x) = x - f^\vee(x)Df = x - f(x)Df^\vee. \tag{1.1.4}$$

By transposition, s_f acts on F: $s_f(g) = g \circ s_f^{-1} = g \circ s_f$. Explicitly, we have

$$s_f(g) = g - <f^\vee, g> f = g - <f, g> f^\vee \tag{1.1.5}$$

for $g \in F$.

For each $u \neq 0$ in V, let $s_u\colon V \to V$ denote the reflection in the hyperplane orthogonal to u, so that

$$s_u(v) = v - <u, v>u^\vee. \tag{1.1.6}$$

Then it is easily checked that

$$Ds_f = s_{Df} \tag{1.1.7}$$

for any non constant $f \in F$.

Let $w : E \to E$ be an isometry. Then w is affine-linear (because it preserves parallelograms) and its derivative Dw is a linear isometry of V, i.e., we have $<(Dw)u, (Dw)v> = <u, v>$ for all $u, v \in V$. The mapping w acts by transposition on F: $(wf)(x) = f(w^{-1}x)$ for $x \in V$, and we have

$$D(wf) = (Dw)(Df). \tag{1.1.8}$$

For each $v \in V$ we shall denote by $t(v) : E \to E$ the translation by v, so that $t(v)x = x + v$. The translations are the isometries of E whose derivative is the identity mapping of V. On F, $t(v)$ acts as follows:

$$t(v)f = f - <Df, v>c \tag{1.1.9}$$

where c is the constant function equal to 1. For if $x \in E$ we have

$$(t(v)f)(x) = f(x - v) = f(x) - <Df, v>.$$

Let w: $E \to E$ be an isometry and let $v \in V$. Then

$$wt(v)w^{-1} = t((Dw)v). \tag{1.1.10}$$

For if $x \in E$ we have

$$(wt(v)w^{-1})(x) = w(w^{-1}x + v) = x + (Dw)v.$$

1.2 Affine root systems

As in §1.1 let E be a real Euclidean space of dimension $n > 0$, and let V be its vector space of translations. We give E the usual topology, defined by the metric $d(x, y) = |x - y|$, so that E is locally compact. As before, let F denote the space (of dimension $n + 1$) of affine-linear functions on E.

An *affine root system* on E [M2] is a subset S of F satisfying the following axioms (AR1)–(AR4):

(AR 1) S spans F, *and the elements of S are non-constant functions.*
(AR 2) $s_a(b) \in S$ *for all $a, b \in S$.*
(AR 3) $<a^\vee, b> \in \mathbb{Z}$ *for all $a, b \in S$.*

The elements of S are called *affine roots*, or just *roots*. Let W_S be the group of isometries of E generated by the reflections s_a for all $a \in S$. This group W_S is the *Weyl group* of S. The fourth axiom is now

(AR 4) *W_S (as a discrete group) acts properly on E.*

In other words, if K_1 and K_2 are compact subsets of E, the set of $w \in W_S$ such that $wK_1 \cap K_2 \neq \emptyset$ is *finite*.

From (AR3) it follows, just as in the case of a finite root system, that if a and λa are proportional affine roots, then λ is one of the numbers $\pm\frac{1}{2}, \pm 1, \pm 2$. If $a \in S$ and $\frac{1}{2}a \notin S$, the root a is said to be *indivisible*. If each $a \in S$ is indivisible, i.e., if the only roots proportional to $a \in S$ are $\pm a$, the root system S is said to be *reduced*.

If S is an affine root system on E, then

$$S^\vee = \{a^\vee : a \in S\}$$

is also an affine root system on E, called the *dual* of S. Clearly S and $S^\vee$ have the same Weyl group, and $S^{\vee\vee} = S$.

The *rank* of S is defined to be the dimension n of E (or V). If S' is another affine root system on a Euclidean space E', an *isomorphism* of S onto S' is a bijection of S onto S' that is induced by an isometry of E onto E'. If S' is isomorphic to λS for some nonzero $\lambda \in \mathbb{R}$, we say that S and S' are *similar*.

We shall assume throughout that S is *irreducible*, i.e. that there exists no partition of S into two non-empty subsets S_1, S_2 such that $<a_1, a_2> = 0$ for all $a_1 \in S_1$ and $a_2 \in S_2$.

The following proposition ([M2], p. 98) provides examples of affine root systems:

(1.2.1) *Let R be an irreducible finite root system spanning a real finite-dimensional vector space V, and let $<u, v>$ be a positive-definite symmetric bilinear form on V, invariant under the Weyl group of R. For each $\alpha \in R$ and $r \in \mathbb{Z}$ let $a_{\alpha,r}$ denote the affine-linear function on V defined by*

$$a_{\alpha,r}(x) = <\alpha, x> + r.$$

Then the set $S(R)$ of functions $a_{\alpha,r}$, where $\alpha \in R$ and r is any integer if $\frac{1}{2}\alpha \notin R$ (resp. any odd integer if $\frac{1}{2}\alpha \in R$) is a reduced irreducible affine root system on V.

Moreover, every reduced irreducible affine root system is similar to either $S(R)$ or $S(R)^\vee$, where R is a finite (but not necessarily reduced) irreducible root system ([M2], §6).

Let S be an irreducible affine root system on a Euclidean space E. The set $\{H_a \colon a \in S\}$ of affine hyperplanes in E on which the affine roots vanish is locally finite ([M2], §4). Hence the set $E - \bigcup_{a \in S} H_a$ is open in E, and therefore so also are the connected components of this set, since E is locally connected. These components are called the *alcoves* of S, or of W_S, and it is a basic fact (loc. cit.) that the Weyl group W_S acts faithfully and transitively on the set of alcoves. Each alcove is an open rectilinear n-simplex, where n is the rank of S.

Choose an alcove C once and for all. Let x_i $(i \in I)$ be the vertices of C, so that C is the set of all points $x = \sum \lambda_i x_i$ such that $\sum \lambda_i = 1$ and each λ_i is a positive real number. Let $B = B(C)$ be the set of indivisible affine roots $a \in S$ such that (i) H_a is a wall of C, and (ii) $a(x) > 0$ for all $x \in C$. Then B consists of $n + 1$ roots, one for each wall of C, and B is a basis of the space F of affine-linear functions on E. The set B is called a *basis* of S.

The elements of B will be denoted by a_i $(i \in I)$, the notation being chosen so that $a_i(x_j) = 0$ if $i \neq j$. Since x_i is in the closure of C, we have $a_i(x_i) > 0$. Moreover, $<a_i, a_j> \leq 0$ whenever $i \neq j$.

The alcove C having been chosen, an affine root $a \in S$ is said to be *positive* (resp. *negative*) if $a(x) > 0$ (resp. $a(x) < 0$) for $x \in C$. Let S^+ (resp. S^-) denote the set of positive (resp. negative) affine roots; then $S = S^+ \cup S^-$ and $S^- = -S^+$. Moreover, each $a \in S^+$ is a linear combination of the a_i with nonnegative integer coefficients, just as in the finite case ([M2], §4).

Let $\alpha_i = Da_i$ $(i \in I)$. The $n + 1$ vectors $\alpha_i \in V$ are linearly dependent, since $\dim V = n$. There is a unique linear relation of the form

$$\sum_{i \in I} m_i \alpha_i = 0$$

where the m_i are positive integers with no common factor, and at least one of the m_i is equal to 1. Hence the function

$$c = \sum_{i \in I} m_i a_i \tag{1.2.2}$$

is constant on E (because its derivative is zero) and positive (because it is positive on C).

Let

$$\Sigma = \{Da : a \in S\}.$$

Then Σ is an irreducible (finite) root system in V. A vertex x_i of the alcove C is said to be *special* for S if (i) $m_i = 1$ and (ii) the vectors α_j ($j \in I$, $j \neq i$) form a basis of Σ. For each affine root system S there is at least one special vertex (see the tables in §1.3). We shall choose a special vertex once and for all, and denote it by x_0 (so that 0 is a distinguished element of the index set I). Thus $m_0 = 1$ in (1.2.2), and if we take x_0 as origin in E, thereby identifying E with V, the affine root a_i ($i \neq 0$) is identified with α_i.

The Cartan matrix and the Dynkin diagram of an irreducible affine root system S are defined exactly as in the finite case. The *Cartan matrix* of S is the matrix $N = (n_{ij})_{i,j \in I}$ where $n_{ij} = \langle a_i^\vee, a_j \rangle$. It has $n+1$ rows and columns, and its rank is n. Its diagonal entries are all equal to 2, and its off-diagonal entries are integers ≤ 0. If $m = (m_i)_{i \in I}$ is the column vector formed by the coefficients in (1.2.2), we have $Nm = 0$.

The *Dynkin diagram* of S is the graph with vertex set I, in which each pair of distinct vertices i, j is joined by d_{ij} edges, where $d_{ij} = \max(|n_{ij}|, |n_{ji}|)$. We have $d_{ij} \leq 4$ in all cases. For each pair of vertices i, j such that $d_{ij} > 0$ and $|a_i| > |a_j|$, we insert an arrowhead (or inequality sign) pointing towards the vertex j corresponding to the shorter root.

If S is reduced, the Dynkin diagram of $S^\vee$ is obtained from that of S by reversing all arrowheads. If $S = S(R)$ as in (1.2.1), where R is irreducible and reduced, the Dynkin diagram of S is the 'completed Dynkin diagram' of R([B1], ch. 6).

If S is reduced, the Cartan matrix and the Dynkin diagram each determine S up to similarity. If S is not reduced, the Dynkin diagram still determines S, provided that the vertices $i \in I$ such that $2a_i \in S$ are marked (e.g. with an asterisk).

1.3 Classification of affine root systems

Let S be an irreducible affine root system. If S is reduced, then S is similar to either $S(R)$ or $S(R)^\vee$ (1.2.1), where R is an irreducible root system. If R is of type X, where X is one of the symbols $A_n, B_n, C_n, D_n, BC_n, E_6, E_7, E_8, F_4, G_2$, we say that $S(R)$ (resp. $S(R)^\vee$) is of type X (resp. $X^\vee$).

If S is not reduced, it determines two reduced affine root systems

$$S_1 = \{a \in S : \tfrac{1}{2}a \notin S\}, \quad S_2 = \{a \in S : 2a \notin S\}$$

with the same affine Weyl group, and $S = S_1 \cup S_2$. We say that S is of type (X, Y) where X, Y are the types of S_1, S_2 respectively.

The reduced and non-reduced irreducible affine root systems are listed below ((1.3.1)–(1.3.18)). In this list, $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of orthonormal vectors in a real Hilbert space.

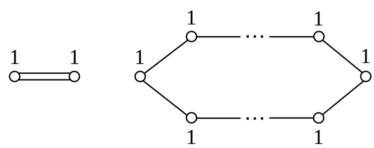
For each type we shall exhibit

(a) an affine root system S of that type;
(b) a basis of S;
(c) the Dynkin diagram of S. Here the numbers attached to the vertices of the diagram are the coefficients m_i in (1.2.2).

We shall first list the reduced systems ((1.3.1)–(1.3.14)) and then the non-reduced systems ((1.3.15)–(1.3.18)).

(1.3.1) *Type* A_n $(n \geq 1)$.

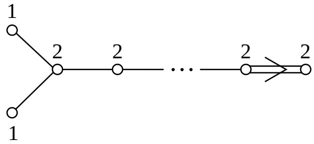
(a) $\pm(\varepsilon_i - \varepsilon_j) + r$ $(1 \leq i < j \leq n+1;\ r \in \mathbb{Z})$.
(b) $a_0 = -\varepsilon_1 + \varepsilon_{n+1} + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq n)$.
(c)



$(n = 1)$ $\qquad$ $(n \geqslant 2)$

(1.3.2) *Type* B_n $(n \geq 3)$.

(a) $\pm\varepsilon_i + r$ $(1 \leq i \leq n;\ r \in \mathbb{Z})$; $\quad \pm\varepsilon_i \pm \varepsilon_j + r$ $(1 \leq i < j \leq n;\ r \in \mathbb{Z})$.
(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq n-1), \quad a_n = \varepsilon_n$.
(c)

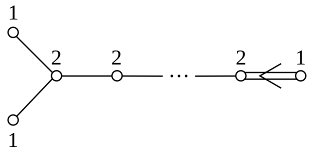


(1.3.3) *Type* $B_n^\vee$ $(n \geq 3)$.

(a) $\pm 2\varepsilon_i + 2r$ $(1 \leq i \leq n;\ r \in \mathbb{Z})$; $\quad \pm\varepsilon_i \pm \varepsilon_j + r$ $(1 \leq i < j \leq n; r \in \mathbb{Z})$.

(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = 2\varepsilon_n.$

(c)

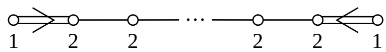


(1.3.4) *Type* C_n ($n \ge 2$).

(a) $\pm 2\varepsilon_i + r\ (1 \le i \le n;\ r \in \mathbb{Z}); \quad \pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n; r \in \mathbb{Z}).$

(b) $a_0 = -2\varepsilon_1 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = 2\varepsilon_n.$

(c)

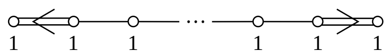


(1.3.5) *Type* $C_n^\vee$ ($n \ge 2$).

(a) $\pm\varepsilon_i + \frac{1}{2}r\ (1 \le i \le n;\ r \in \mathbb{Z}); \quad \pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z}).$

(b) $a_0 = -\varepsilon_1 + \frac{1}{2}, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_n.$

(c)



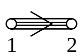
(1.3.6) *Type* BC_n ($n \ge 1$).

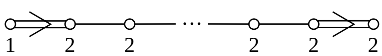
(a) $\pm\varepsilon_i + r\ (1 \le i \le n;\ r \in \mathbb{Z}); \quad \pm 2\varepsilon_i + 2r + 1\ (1 \le i \le n;\ r \in \mathbb{Z});$

$$\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z}).$$

(b) $a_0 = -2\varepsilon_1 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_n.$

(c)





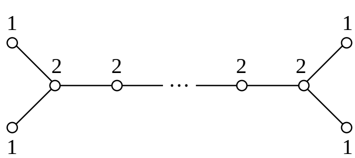
($n = 1$) ($n \geqslant 2$)

(1.3.7) *Type* D_n ($n \ge 4$).

(a) $\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z})$

(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_{n-1} + \varepsilon_n.$

(c)



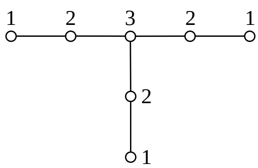
These are the "classical" reduced affine root systems. The next seven types ((1.3.8)–(1.3.14)) are the "exceptional" reduced affine root systems. In (1.3.8)–(1.3.10) let

$$\omega_i = \varepsilon_i - \frac{1}{9}(\varepsilon_1 + \cdots + \varepsilon_9) \qquad (1 \le i \le 9).$$

(1.3.8) *Type* E_6.

(a) $\pm(\omega_i - \omega_j) + r\ (1 \le i < j \le 6;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_j + \omega_k) + r\ (1 \le i < j < k \le 6;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_2 + \cdots + \omega_6) + r\ (r \in \mathbb{Z})$.

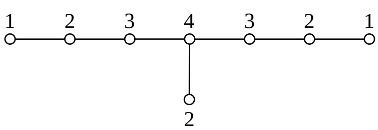
(b) $a_0 = -(\omega_1 + \cdots + \omega_6) + 1, \quad a_i = \omega_i - \omega_{i+1}\ (1 \le i \le 5)$,
$a_6 = \omega_4 + \omega_5 + \omega_6$.

(c) 

(1.3.9) *Type* E_7.

(a) $\pm(\omega_i - \omega_j) + r\ (1 \le i < j \le 7;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_j + \omega_k) + r\ (1 \le i < j < k \le 7;\ r \in \mathbb{Z})$;
$\pm(\omega_1 + \cdots + \hat{\omega}_i + \cdots + \omega_7) + r\ (1 \le i \le 7;\ r \in \mathbb{Z})$.

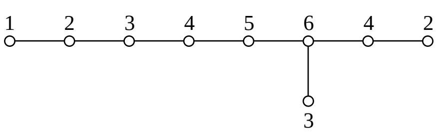
(b) $a_0 = -(\omega_1 + \cdots + \omega_6) + 1, \quad a_i = \omega_i - \omega_{i+1}\ (1 \le i \le 6)$,
$a_7 = \omega_5 + \omega_6 + \omega_7$.

(c) 

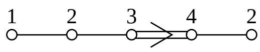
(1.3.10) *Type* E_8.

(a) $\pm(\omega_i - \omega_j) + r\ (1 \le i < j \le 9;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_j + \omega_k) + r\ (1 \le i < j < k \le 9;\ r \in \mathbb{Z})$.

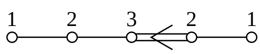
(b) $a_0 = \omega_1 - \omega_2 + 1, \quad a_i = \omega_{i+1} - \omega_{i+2}\ (1 \le i \le 7)$,
$a_8 = \omega_7 + \omega_8 + \omega_9$.

(c) 

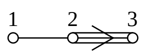
(1.3.11) *Type* F_4.

(a) $\pm\varepsilon_i + r$ $(1 \le i \le 4;\ r \in \mathbb{Z})$; $\pm\varepsilon_i \pm \varepsilon_j + r$ $(1 \le i < j \le 4;\ r \in \mathbb{Z})$;
$\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) + r$ $(r \in \mathbb{Z})$.
(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = \varepsilon_3 - \varepsilon_4$, $a_3 = \varepsilon_4$,
$a_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$.
(c) 

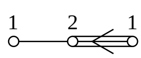
(1.3.12) *Type* $F_4^\vee$.

(a) $\pm 2\varepsilon_i + 2r$ $(1 \le i \le 4;\ r \in \mathbb{Z})$; $\pm\varepsilon_i \pm \varepsilon_j + r$ $(1 \le i < j \le 4;\ r \in \mathbb{Z})$;
$\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 + 2r$ $(r \in \mathbb{Z})$.
(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = \varepsilon_3 - \varepsilon_4$, $a_3 = 2\varepsilon_4$,
$a_4 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4$.
(c) 

(1.3.13) *Type* G_2.

(a) $\pm(\varepsilon_i - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)) + r$ $(1 \le i \le 3;\ r \in \mathbb{Z})$;
$\pm(\varepsilon_i - \varepsilon_j) + r$ $(1 \le i < j \le 3;\ r \in \mathbb{Z})$.
(b) $a_0 = \varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = \varepsilon_3 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$.
(c) 

(1.3.14) *Type* $G_2^\vee$.

(a) $\pm(3\varepsilon_i - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)) + 3r$ $(1 \le i \le 3;\ r \in \mathbb{Z})$;
$\pm(\varepsilon_i - \varepsilon_j) + r$ $(1 \le i < j \le 3;\ r \in \mathbb{Z})$.
(b) $a_0 = \varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = 3\varepsilon_3 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$.
(c) 

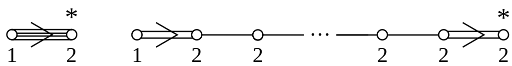
We come now to the non-reduced affine root systems. In the Dynkin diagrams below, an asterisk placed over a vertex indicates that if a_i is the affine root corresponding to that vertex in a basis of S, then $2a_i \in S$.

(1.3.15) *Type* (BC_n, C_n) $(n \ge 1)$.

(a) $\pm\varepsilon_i + r$, $\pm 2\varepsilon_i + r$ $(1 \le i \le n, r \in \mathbb{Z})$;
$\pm\varepsilon_i \pm \varepsilon_j + r$ $(1 \le i < j \le n;\ r \in \mathbb{Z})$.

(b) $a_0 = -2\varepsilon_1 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_n.$

(c)



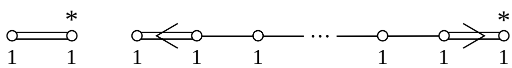
$(n = 1)$ $(n \geqslant 2)$

(1.3.16) *Type* $(C_n^\vee, BC_n)$ $(n \ge 1)$.

(a) $\pm\varepsilon_i + \frac{1}{2}r, \quad \pm 2\varepsilon_i + 2r\ (1 \le i \le n;\ r \in \mathbb{Z});$
$\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z}).$

(b) $a_0 = -\varepsilon_1 + \frac{1}{2}, \quad a_i = \varepsilon_i - \varepsilon_{i+1}, \quad a_n = \varepsilon_n.$

(c)



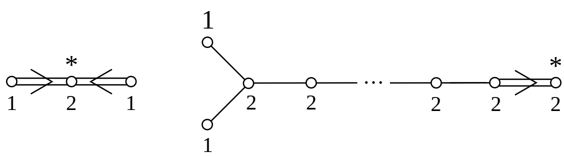
$(n = 1)$ $(n \geqslant 2)$

(1.3.17) *Type* $(C_2, C_2^\vee), (B_n, B_n^\vee)$ $(n \ge 3)$.

(a) $\pm\varepsilon_i + r, \quad \pm 2\varepsilon_i + 2r\ (1 \le i \le n;\ r \in \mathbb{Z});$
$\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z}).$

(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1); \quad a_n = \varepsilon_n.$

(c)



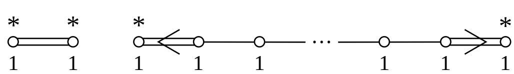
$(n = 2)$ $(n \geqslant 3)$

(1.3.18) *Type* $(C_n^\vee, C_n)$ $(n \ge 1)$.

(a) $\pm\varepsilon_i + \frac{1}{2}r, \quad \pm 2\varepsilon_i + r\ (1 \le i \le n;\ r \in \mathbb{Z});$
$\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z}).$

(b) $a_0 = -\varepsilon_1 + \frac{1}{2}, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_n.$

(c)



$(n = 1)$ $(n \geqslant 2)$

For each irreducible affine root system S, let $o(S)$ denote the number of W_S-orbits in S. If S is reduced, the list above shows that $o(S) \le 3$, and that $o(S) = 3$

only when S is of type C_n, $C_n^\vee$ or $BC_n (n \geq 2)$. If S is not reduced, the maximum value of $o(S)$ is 5, and is attained only when S is of type $(C_n^\vee, C_n)$ $(n \geq 2)$. The five orbits are $O_1, \ldots, O_5$ where, in the notation of (1.3.18) above,

$$O_1 = \{\pm\varepsilon_i + r : 1 \leq i \leq n, r \in \mathbb{Z}\}, \quad O_2 = 2O_1, \quad O_3 = O_1 + \tfrac{1}{2},$$
$$O_4 = 2O_3 = O_2 + 1, \quad O_5 = \{\pm\varepsilon_i \pm \varepsilon_j + r : 1 \leq i < j \leq n, \ r \in \mathbb{Z}\}.$$

Finally, the list above shows that all the non-reduced irreducible affine root systems of rank n are subsystems of (1.3.18), obtained by deleting one or more of the W_S-orbits; and so are the "classical" root systems (1.3.2)–(1.3.7).

1.4 Duality

In later chapters, in order to formulate conveniently certain dualities, we shall need to consider not one but a pair (S, S') of irreducible affine root systems, together with a pair (R, R') of finite root systems and a pair (L, L') of lattices in V.

Let R be a reduced finite irreducible root system in V, and let P (resp. $P^\vee$) denote the weight lattice of R (resp. $R^\vee$), and Q (resp. $Q^\vee$) the root lattice of R (resp. $R^\vee$). Fix a basis $(\alpha_i)_{i \in I_0}$ of R, and let φ be the highest root of R relative to this basis. In (1.4.1) and (1.4.2) below we shall assume that the scalar product on V is normalized so that $|\varphi|^2 = 2$ and therefore $\varphi^\vee = \varphi$. (This conflicts with standard usage, as in § 1.3, only when R is of type C_n (1.3.4).)

The pairs (S, S'), (R, R'), (L, L') to be considered are the following:

(1.4.1) $\quad S = S(R), \quad S' = S(R^\vee); \quad R' = R^\vee; \quad L = P, \quad L' = P^\vee.$

Then S (resp. S') has a basis $(a_i)_{i \in I}$ (resp. $(a'_i)_{i \in I}$) where $a_i = \alpha_i$ $(i \neq 0)$, $a_0 = -\varphi + c$; $a'_i = \alpha_i^\vee$ $(i \neq 0)$, $a'_0 = -\psi^\vee + c$, where ψ is the highest *short* root of R.

(1.4.2) $\quad S = S' = S(R)^\vee; \quad R' = R; \quad L = L' = P^\vee.$

Then $S = S'$ has a basis $(a_i)_{i \in I} = (a'_i)_{i \in I}$, where $a_i = a'_i = \alpha_i^\vee$ if $i \neq 0$, and $a_0 = a'_0 = -\varphi + c$.

(1.4.3) $\quad S = S'$ is of type $(C_n^\vee, C_n)$; $R = R'$ is of type C_n; $L = L' = Q^\vee$. We shall assume that S is as given in (1.3.18), so that $a_i = \alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq n-1)$ and $\alpha_n = 2a_n = 2\varepsilon_n$, and $L = \mathbb{Z}^n$.

For each $\alpha \in R$, let $\alpha' (= \alpha$ or $\alpha^\vee)$ be the corresponding element of R'. Then $\langle\lambda', \alpha\rangle$ and $\langle\lambda, \alpha'\rangle$ are integers, for all $\lambda \in L$, $\lambda' \in L'$ and $\alpha \in R$.

In each case let

$$\Omega' = L'/Q^\vee, \tag{1.4.4}$$

a finite abelian group. Also let

$$<L, L'> = \{<\lambda, \lambda'> : \lambda \in L, \lambda' \in L'\}.$$

Then we have

$$<L, L'> = e^{-1}\mathbb{Z} \tag{1.4.5}$$

where e is the exponent of Ω', except in case (1.4.2) when R is of type B_n or C_{2n}, in which case $e = 1$.

Anticipating Chapter 2, let $W = W(R, L')$ be the group of displacements of V generated by the Weyl group W_0 of R and the translations $t(\lambda')$, $\lambda' \in L'$, so that W is the semidirect product of W_0 and $t(L')$:

$$W = W(R, L') = W_0 \ltimes t(L'). \tag{1.4.6}$$

Dually, let

$$W' = W(R', L) = W_0 \ltimes t(L). \tag{1.4.6$'$}$$

By transposition, both W and W' act on F.

(1.4.7) *W permutes S and W' permutes S'.*

This follows from the fact, remarked above, that $<\lambda', \alpha>$ and $<\lambda, \alpha'>$ are integers, for all $\lambda \in L$, $\lambda' \in L'$ and $\alpha \in R$.

Now let

$$\Lambda = L \oplus \mathbb{Z}c_0 \tag{1.4.8}$$

where $c_0 = e^{-1}c$. We shall regard elements of Λ as functions on V: if $f \in \Lambda$, say $f = \lambda + rc_0$ where $\lambda \in L$ and $r \in \mathbb{Z}$, then

$$f(x) = <\lambda, x> + e^{-1}r$$

for $x \in V$. Then Λ is a lattice in F.

(1.4.9) *Λ is stable under the action of W.*

Proof Let $w \in W$, say $w = vt(\lambda')$ where $v \in W_0$ and $\lambda' \in L'$. If $f = \lambda + rc_0 \in \Lambda$ and $x \in V$, we have

$$\begin{aligned} wf(x) &= f(w^{-1}x) = f(v^{-1}x - \lambda') \\ &= <\lambda, v^{-1}x - \lambda'> + e^{-1}r \\ &= <v\lambda, x> + e^{-1}r - <\lambda, \lambda'> \end{aligned}$$

so that

$$wf = v\lambda + (r - e<\lambda, \lambda'>)c_0$$

is in Λ, since $e<\lambda, \lambda'> \in \mathbb{Z}$ by (1.4.5). □

1.5 Labels

Let S be an irreducible affine root system as in §1.4 and let $W = W(R, L')$. A *W-labelling* k of S is a mapping $k : S \to \mathbb{R}$ such that $k(a) = k(b)$ if a, b are in the same W-orbit in S.

If $S = S(R)$ where R is simply-laced (types A, D, E), all the labels $k(a)$ are equal. If $S = S(R)$ or $S(R)^\vee$ where $R \neq R^\vee$, there are at most two labels, one for short roots and one for long roots. Finally, if S is of type $(C_n^\vee, C_n)$ as in (1.4.3), there are five W-orbits $O_1, \dots, O_5$ in S, as observed in §1.3, and correspondingly five labels $k_1, \dots, k_5$, where $k_i = k(a)$ for $a \in O_i$.

Given a labelling k of S as above, we define a *dual labelling* k' of S', as follows:

(a) if $S = S(R)$, $S' = S(R^\vee)$ (1.4.1) and $a' = \alpha^\vee + rc \in S'$, then $k'(a') = k(\alpha + rc)$.

(b) If $S = S' = S(R)^\vee$ (1.4.2), then $k' = k$.

(c) If $S = S'$ is of type $(C_n^\vee, C_n)$ (1.4.3), the dual labels k'_i $(1 \leq i \leq 5)$ are defined by

(1.5.1)
$$\begin{aligned} k'_1 &= \tfrac{1}{2}(k_1 + k_2 + k_3 + k_4), \\ k'_2 &= \tfrac{1}{2}(k_1 + k_2 - k_3 - k_4), \\ k'_3 &= \tfrac{1}{2}(k_1 - k_2 + k_3 - k_4), \\ k'_4 &= \tfrac{1}{2}(k_1 - k_2 - k_3 + k_4), \\ k'_5 &= k_5, \end{aligned}$$

and $k'(a) = k'_i$ if $a \in O_i$.

In all cases let

$$\rho_{k'} = \frac{1}{2}\sum_{\alpha\in R^+} k'(\alpha^\vee)\alpha,$$

(1.5.2)

$$\rho'_k = \frac{1}{2}\sum_{\alpha\in R^+} k(\alpha'^\vee)\alpha'.$$

where R^+ is the set of positive roots of R determined by the basis (α_i). Explicitly, when $S = S(R)$ (1.4.1) we have

$$\rho_{k'} = \frac{1}{2}\sum_{\alpha\in R^+} k(\alpha)\alpha,$$
$$\rho'_k = \frac{1}{2}\sum_{\alpha\in R^+} k(\alpha)\alpha^\vee;$$

when $S = S(R)^\vee$ (1.4.2) we have

$$\rho_{k'} = \rho'_k = \frac{1}{2}\sum_{\alpha\in R^+} k(\alpha^\vee)\alpha;$$

and when S is of type $(C_n^\vee, C_n)$ (1.4.3), so that R is of type C_n,

$$\rho_{k'} = \sum_{i=1}^{n}(k'_1 + (n-i)k_5)\varepsilon_i,$$
$$\rho'_k = \sum_{i=1}^{n}(k_1 + (n-i)k_5)\varepsilon_i.$$

For each $w \in W_0$, we have

$$w^{-1}\rho'_k = \frac{1}{2}\sum_{\alpha\in R^+} \sigma(w\alpha)k(\alpha'^\vee)\alpha',$$

(1.5.3)

$$w^{-1}\rho_{k'} = \frac{1}{2}\sum_{\alpha\in R^+} \sigma(w\alpha)k'(\alpha^\vee)\alpha,$$

where $\sigma(w\alpha) = +1$ or -1 according as $w\alpha \in R^+$ or R^-. In particular, if $i \in I$, $i \neq 0$ we have

$$s_i\rho'_k = \rho'_k - k\big(\alpha_i'^\vee\big)\alpha'_i,$$

(1.5.4)

$$s_i\rho_{k'} = \rho_{k'} - k'\big(\alpha_i^\vee\big)\alpha_i.$$

(1.5.5) *If the labels $k(\alpha_i'^\vee)$, $k'(\alpha_i^\vee)$ are all nonzero, then ρ'_k and $\rho_{k'}$ are fixed only by the identity element of W_0.* □

Notes and references

Affine root systems were introduced in [M2], which contains an account of their basic properties and their classification. The list of Dynkin diagrams in §1.3 will also be found in the article of Bruhat and Tits [B3] (except that both [M2] and [B3] omit the diagram (1.3.17) when $n = 2$). The reduced affine root systems (1.3.1)–(1.3.14) are in one-one correspondence with the irreducible affine (or Euclidean) Kac-Moody Lie algebras, and correspondingly their diagrams appear in Moody's paper [M9] and Kac's book [K1].