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157 Affine Hecke Algebras and Orthogonal Polynomials

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Introduction

Over the last fifteen years or so, there has emerged a satisfactory and coherent theory of orthogonal polynomials in several variables, attached to root systems, and depending on two or more parameters. At the present stage of its development, it appears that an appropriate framework for its study is provided by the notion of an affine root system: to each irreducible affine root system S there are associated several families of orthogonal polynomials (denoted by E_λ, P_λ, Q_λ, $P_\lambda^{(\varepsilon)}$ in this book). For example, when S is the non-reduced affine root system of rank 1 denoted here by $(C_1^\vee, C_1)$, the polynomials P_λ are the Askey-Wilson polynomials [A2] which, as is well-known, include as special or limiting cases *all* the classical families of orthogonal polynomials in one variable.

I have surveyed elsewhere [M8] the various antecedents of this theory: symmetric functions, especially Schur functions and their generalizations such as zonal polynomials and Hall-Littlewood functions [M6]; zonal spherical functions on p-adic Lie groups [M1]; the Jacobi polynomials of Heckman and Opdam attached to root systems [H2]; and the constant term conjectures of Dyson, Andrews et al. ([D1], [A1], [M4], [M10]). The lectures of Kirillov [K2] also provide valuable background and form an excellent introduction to the subject.

The title of this monograph is the same as that of the lecture [M7]. That report, for obvious reasons of time and space, gave only a cursory and incomplete overview of the theory. The modest aim of the present volume is to fill in the gaps in that report and to provide a unified foundation for the theory in its present state.

The decision to treat all affine root systems, reduced or not, simultaneously on the same footing has resulted in an unavoidably complex system of notation. In order to formulate results uniformly it is necessary to associate to each affine root system S another affine root system S' (which may or may not coincide with S), and to each labelling (§1.5) of S a dual labelling of S'.

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Frontmatter
More information

The prospective reader is expected to be familiar with the algebra and geometry of (crystallographic) root systems and Weyl groups, as expounded for example by Bourbaki in [B1]. Beyond that, the book is pretty well self-contained.

We shall now survey briefly the various chapters and their contents. The first four chapters are preparatory to Chapter 5, which contains all the main results. Chapter 1 covers the basic properties of affine root systems and their classification. Chapter 2 is devoted to the extended affine Weyl group, and collects various notions and results that will be needed later.

Chapter 3 introduces the (Artin) braid group of an extended affine Weyl group, and the double braid group. The main result of this chapter is the duality theorem (3.5.1); although it is fundamental to the theory, there is at this time of writing no complete proof in the literature. I have to confess that the proof given here of the duality theorem is the least satisfactory feature of the book, since it consists in checking, in rather tedious detail, the necessary relations between the generators. Fortunately, B. Ion [I1] has recently given a more conceptual proof which avoids these calculations.

The subject of Chapter 4 is the affine Hecke algebra $\mathfrak{H}$, which is a deformation of the group algebra of the extended affine Weyl group. We construct the basic representation of $\mathfrak{H}$ in §4.3 and develop its properties in the subsequent sections. Finally, in §4.7 we introduce the double affine Hecke algebra $\tilde{\mathfrak{H}}$, and show that the duality theorem for the double braid group gives rise to a duality theorem for $\tilde{\mathfrak{H}}$.

As stated above, Chapter 5, on orthogonal polynomials, is the heart of the book. The scalar products are introduced in §5.1, the orthogonal polynomials E_λ in §5.2, the symmetric orthogonal polynomials P_λ in §5.3, and their variants Q_λ and $P_\lambda^{(\varepsilon)}$ in §5.7. The main results of the chapter are the symmetry theorems (5.2.4) and (5.3.5); the specialization theorems (5.2.14) and (5.3.12); and the norm formulas (5.8.17) and (5.8.19), which include as special cases almost all the constant term conjectures referred to earlier.

The final Chapter 6 deals with the case where the affine root system S has rank 1. Here everything can be made completely explicit. When S is of type A_1, the polynomials P_λ are the continuous q-ultraspherical (or Rogers) polynomials, and when S is of type $(C_1^\vee, C_1)$ they are the Askey-Wilson polynomials, as mentioned above.

The subject of this monograph has many connections with other parts of mathematics and theoretical physics, such as (in no particular order) algebraic combinatorics, harmonic analysis, integrable quantum systems, quantum groups and symmetric spaces, quantum statistical mechanics, deformed

Virasoro algebras, and string theory. I have made no attempt to survey these various applications, partly from lack of competence, but also because an adequate account would require a book of its own.

Finally, references to the history and the literature will be found in the Notes and References at the end of each chapter.