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Excerpt
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Affine root systems

1.1 Notation and terminology

Let E be an affine space over a field K: that is to say, E is a set on which a K-vector space V acts faithfully and transitively. The elements of V are called *translations* of E, and the effect of a translation $v \in V$ on $x \in E$ is written $x + v$. If $y = x + v$ we write $v = y - x$.

Let E' be another affine space over K, and let V' be its vector space of translations. A mapping $f : E \to E'$ is said to be *affine-linear* if there exists a K-linear mapping $Df : V \to V'$, called the *derivative* of f, such that

$$f(x + v) = f(x) + (Df)(v). \qquad (1.1.1)$$

for all $x \in E$ and $v \in V$. In particular, a function $f : E \to K$ is affine-linear if and only if there exists a linear form $Df : V \to K$ such that (1.1.1) holds.

If $f, g : E \to K$ are affine-linear and $\lambda, \mu \in K$, the function $h = \lambda f + \mu g : x \mapsto \lambda f(x) + \mu g(x)$ is affine-linear, with derivative $Dh = \lambda Df + \mu Dg$. Hence the set F of all affine-linear functions $f : E \to K$ is a K-vector space, and D is a K-linear mapping of F onto the dual V^* of the vector space V. The kernel of D is the 1-dimensional subspace F^0 of F consisting of the constant functions.

Let F^* be the dual of the vector space F. For each $x \in E$, the evaluation map $\varepsilon_x : f \mapsto f(x)$ belongs to F^*, and the mapping $x \mapsto \varepsilon_x$ embeds E in F^* as an affine hyperplane. Likewise, for each $v \in V$ let $\varepsilon_v \in F^*$ be the mapping $f \mapsto (Df)(v)$. If $v = y - x$, where $x, y \in E$, we have $\varepsilon_v = \varepsilon_y - \varepsilon_x$ by (1.1.1), and the mapping $v \mapsto \varepsilon_v$ embeds V in F^* as the hyperplane through the origin parallel to E.

From now on, K will be the field $\mathbb{R}$ of real numbers, and V will be a real vector space of finite dimension $n > 0$, equipped with a positive definite symmetric scalar product $<u, v>$. We shall write

$$|v| = <v, v>^{1/2}$$

for the length of a vector $v \in V$. Then E is a Euclidean space of dimension n, and is a metric space for the distance function $d(x, y) = |x - y|$.

We shall identify V with its dual space V^* by means of the scalar product $<u, v>$. For any affine-linear function $f: E \to \mathbb{R}$, (1.1.1) now takes the form

(1.1.2) $$f(x + v) = f(x) + <Df, v>$$

and Df is the *gradient* of f, in the usual sense of calculus.

We define a scalar product on the space F as follows:

(1.1.3) $$<f, g> = <Df, Dg>.$$

This scalar product is positive semidefinite, with radical the one-dimensional space F^0 of constant functions.

For each $v \neq 0$ in V let

$$v^\vee = 2v/|v|^2$$

and for each non-constant $f \in F$ let

$$f^\vee = 2f/|f|^2.$$

Also let

$$H_f = f^{-1}(0)$$

which is an affine hyperplane in E. The reflection in this hyperplane is the isometry $s_f: E \to E$ given by the formula

(1.1.4) $$s_f(x) = x - f^\vee(x)Df = x - f(x)Df^\vee.$$

By transposition, s_f acts on F: $s_f(g) = g \circ s_f^{-1} = g \circ s_f$. Explicitly, we have

(1.1.5) $$s_f(g) = g - <f^\vee, g>f = g - <f, g>f^\vee$$

for $g \in F$.

For each $u \neq 0$ in V, let $s_u: V \to V$ denote the reflection in the hyperplane orthogonal to u, so that

(1.1.6) $$s_u(v) = v - <u, v>u^\vee.$$

Then it is easily checked that

(1.1.7) $$Ds_f = s_{Df}$$

for any non constant $f \in F$.

Let $w : E \to E$ be an isometry. Then w is affine-linear (because it preserves parallelograms) and its derivative Dw is a linear isometry of V, i.e., we have $<(Dw)u, (Dw)v> = <u, v>$ for all $u, v \in V$. The mapping w acts by transposition on F: $(wf)(x) = f(w^{-1}x)$ for $x \in V$, and we have

(1.1.8) $$D(wf) = (Dw)(Df).$$

For each $v \in V$ we shall denote by $t(v) : E \to E$ the translation by v, so that $t(v)x = x + v$. The translations are the isometries of E whose derivative is the identity mapping of V. On F, $t(v)$ acts as follows:

(1.1.9) $$t(v)f = f - <Df, v>c$$

where c is the constant function equal to 1. For if $x \in E$ we have

$$(t(v)f)(x) = f(x - v) = f(x) - <Df, v>.$$

Let w: $E \to E$ be an isometry and let $v \in V$. Then

(1.1.10) $$wt(v)w^{-1} = t((Dw)v).$$

For if $x \in E$ we have

$$(wt(v)w^{-1})(x) = w(w^{-1}x + v) = x + (Dw)v.$$

1.2 Affine root systems

As in §1.1 let E be a real Euclidean space of dimension $n > 0$, and let V be its vector space of translations. We give E the usual topology, defined by the metric $d(x, y) = |x - y|$, so that E is locally compact. As before, let F denote the space (of dimension $n + 1$) of affine-linear functions on E.

An *affine root system* on E [M2] is a subset S of F satisfying the following axioms (AR1)–(AR4):

(AR 1) S spans F, *and the elements of* S *are non-constant functions.*
(AR 2) $s_a(b) \in S$ *for all* $a, b \in S$.
(AR 3) $<a^\vee, b> \in \mathbb{Z}$ *for all* $a, b \in S$.

The elements of S are called *affine roots*, or just *roots*. Let W_S be the group of isometries of E generated by the reflections s_a for all $a \in S$. This group W_S is the *Weyl group* of S. The fourth axiom is now

(AR 4) *W_S (as a discrete group) acts properly on E.*

In other words, if K_1 and K_2 are compact subsets of E, the set of $w \in W_S$ such that $wK_1 \cap K_2 \neq \emptyset$ is *finite*.

From (AR3) it follows, just as in the case of a finite root system, that if a and λa are proportional affine roots, then λ is one of the numbers $\pm\frac{1}{2}, \pm 1, \pm 2$. If $a \in S$ and $\frac{1}{2}a \notin S$, the root a is said to be *indivisible*. If each $a \in S$ is indivisible, i.e., if the only roots proportional to $a \in S$ are $\pm a$, the root system S is said to be *reduced*.

If S is an affine root system on E, then

$$S^\vee = \{a^\vee : a \in S\}$$

is also an affine root system on E, called the *dual* of S. Clearly S and $S^\vee$ have the same Weyl group, and $S^{\vee\vee} = S$.

The *rank* of S is defined to be the dimension n of E (or V). If S' is another affine root system on a Euclidean space E', an *isomorphism* of S onto S' is a bijection of S onto S' that is induced by an isometry of E onto E'. If S' is isomorphic to λS for some nonzero $\lambda \in \mathbb{R}$, we say that S and S' are *similar*.

We shall assume throughout that S is *irreducible*, i.e. that there exists no partition of S into two non-empty subsets S_1, S_2 such that $<a_1, a_2> = 0$ for all $a_1 \in S_1$ and $a_2 \in S_2$.

The following proposition ([M2], p. 98) provides examples of affine root systems:

(1.2.1) *Let R be an irreducible finite root system spanning a real finite-dimensional vector space V, and let $<u, v>$ be a positive-definite symmetric bilinear form on V, invariant under the Weyl group of R. For each $\alpha \in R$ and $r \in \mathbb{Z}$ let $a_{\alpha,r}$ denote the affine-linear function on V defined by*

$$a_{\alpha,r}(x) = <\alpha, x> + r.$$

Then the set $S(R)$ of functions $a_{\alpha,r}$, where $\alpha \in R$ and r is any integer if $\frac{1}{2}\alpha \notin R$ (resp. any odd integer if $\frac{1}{2}\alpha \in R$) is a reduced irreducible affine root system on V.

Moreover, every reduced irreducible affine root system is similar to either $S(R)$ or $S(R)^\vee$, where R is a finite (but not necessarily reduced) irreducible root system ([M2], §6).

Let S be an irreducible affine root system on a Euclidean space E. The set $\{H_a : a \in S\}$ of affine hyperplanes in E on which the affine roots vanish is locally finite ([M2], §4). Hence the set $E - \bigcup_{a\in S} H_a$ is open in E, and therefore so also are the connected components of this set, since E is locally connected. These components are called the *alcoves* of S, or of W_S, and it is a basic fact (loc. cit.) that the Weyl group W_S acts faithfully and transitively on the set of alcoves. Each alcove is an open rectilinear n-simplex, where n is the rank of S.

Choose an alcove C once and for all. Let x_i $(i \in I)$ be the vertices of C, so that C is the set of all points $x = \sum \lambda_i x_i$ such that $\sum \lambda_i = 1$ and each λ_i is a positive real number. Let $B = B(C)$ be the set of indivisible affine roots $a \in S$ such that (i) H_a is a wall of C, and (ii) $a(x) > 0$ for all $x \in C$. Then B consists of $n + 1$ roots, one for each wall of C, and B is a basis of the space F of affine-linear functions on E. The set B is called a *basis* of S.

The elements of B will be denoted by a_i $(i \in I)$, the notation being chosen so that $a_i(x_j) = 0$ if $i \neq j$. Since x_i is in the closure of C, we have $a_i(x_i) > 0$. Moreover, $\langle a_i, a_j \rangle \leq 0$ whenever $i \neq j$.

The alcove C having been chosen, an affine root $a \in S$ is said to be *positive* (resp. *negative*) if $a(x) > 0$ (resp. $a(x) < 0$) for $x \in C$. Let S^+ (resp. S^-) denote the set of positive (resp. negative) affine roots; then $S = S^+ \cup S^-$ and $S^- = -S^+$. Moreover, each $a \in S^+$ is a linear combination of the a_i with nonnegative integer coefficients, just as in the finite case ([M2], §4).

Let $\alpha_i = Da_i$ $(i \in I)$. The $n + 1$ vectors $\alpha_i \in V$ are linearly dependent, since $\dim V = n$. There is a unique linear relation of the form

$$\sum_{i\in I} m_i \alpha_i = 0$$

where the m_i are positive integers with no common factor, and at least one of the m_i is equal to 1. Hence the function

(1.2.2) $$c = \sum_{i\in I} m_i a_i$$

is constant on E (because its derivative is zero) and positive (because it is positive on C).

Let

$$\Sigma = \{Da : a \in S\}.$$

Then Σ is an irreducible (finite) root system in V. A vertex x_i of the alcove C is said to be *special* for S if (i) $m_i = 1$ and (ii) the vectors α_j $(j \in I, j \neq i)$ form a basis of Σ. For each affine root system S there is at least one special vertex (see the tables in §1.3). We shall choose a special vertex once and for all, and denote it by x_0 (so that 0 is a distinguished element of the index set I). Thus $m_0 = 1$ in (1.2.2), and if we take x_0 as origin in E, thereby identifying E with V, the affine root a_i $(i \neq 0)$ is identified with α_i.

The Cartan matrix and the Dynkin diagram of an irreducible affine root system S are defined exactly as in the finite case. The *Cartan matrix* of S is the matrix $N = (n_{ij})_{i,j \in I}$ where $n_{ij} = \langle a_i^\vee, a_j \rangle$. It has $n + 1$ rows and columns, and its rank is n. Its diagonal entries are all equal to 2, and its off-diagonal entries are integers ≤ 0. If $m = (m_i)_{i \in I}$ is the column vector formed by the coefficients in (1.2.2), we have $Nm = 0$.

The *Dynkin diagram* of S is the graph with vertex set I, in which each pair of distinct vertices i, j is joined by d_{ij} edges, where $d_{ij} = \max(|n_{ij}|, |n_{ji}|)$. We have $d_{ij} \leq 4$ in all cases. For each pair of vertices i, j such that $d_{ij} > 0$ and $|a_i| > |a_j|$, we insert an arrowhead (or inequality sign) pointing towards the vertex j corresponding to the shorter root.

If S is reduced, the Dynkin diagram of $S^\vee$ is obtained from that of S by reversing all arrowheads. If $S = S(R)$ as in (1.2.1), where R is irreducible and reduced, the Dynkin diagram of S is the 'completed Dynkin diagram' of R([B1], ch. 6).

If S is reduced, the Cartan matrix and the Dynkin diagram each determine S up to similarity. If S is not reduced, the Dynkin diagram still determines S, provided that the vertices $i \in I$ such that $2a_i \in S$ are marked (e.g. with an asterisk).

1.3 Classification of affine root systems

Let S be an irreducible affine root system. If S is reduced, then S is similar to either $S(R)$ or $S(R)^\vee$ (1.2.1), where R is an irreducible root system. If R is of type X, where X is one of the symbols $A_n, B_n, C_n, D_n, BC_n, E_6, E_7, E_8, F_4, G_2$, we say that $S(R)$ (resp. $S(R)^\vee$) is of type X (resp. $X^\vee$).

If S is not reduced, it determines two reduced affine root systems

$$S_1 = \{a \in S : \tfrac{1}{2}a \notin S\}, \quad S_2 = \{a \in S : 2a \notin S\}$$

with the same affine Weyl group, and $S = S_1 \cup S_2$. We say that S is of type (X, Y) where X, Y are the types of S_1, S_2 respectively.

The reduced and non-reduced irreducible affine root systems are listed below ((1.3.1)–(1.3.18)). In this list, $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of orthonormal vectors in a real Hilbert space.

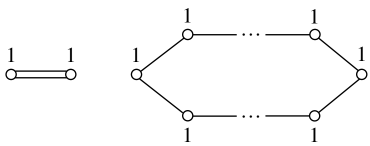
For each type we shall exhibit

(a) an affine root system S of that type;
(b) a basis of S;
(c) the Dynkin diagram of S. Here the numbers attached to the vertices of the diagram are the coefficients m_i in (1.2.2).

We shall first list the reduced systems ((1.3.1)–(1.3.14)) and then the non-reduced systems ((1.3.15)–(1.3.18)).

(1.3.1) *Type* A_n $(n \geq 1)$.

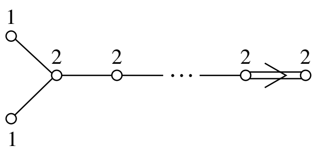
(a) $\pm(\varepsilon_i - \varepsilon_j) + r$ $(1 \leq i < j \leq n+1;\ r \in \mathbb{Z})$.
(b) $a_0 = -\varepsilon_1 + \varepsilon_{n+1} + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \leq i \leq n)$.
(c)



$(n = 1)$ $(n \geqslant 2)$

(1.3.2) *Type* B_n $(n \geq 3)$.

(a) $\pm\varepsilon_i + r$ $(1 \leq i \leq n;\ r \in \mathbb{Z})$; $\pm\varepsilon_i \pm \varepsilon_j + r$ $(1 \leq i < j \leq n;\ r \in \mathbb{Z})$.
(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \leq i \leq n-1), \quad a_n = \varepsilon_n$.
(c)

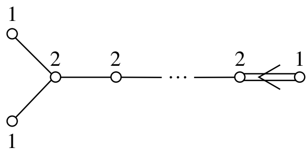


(1.3.3) *Type* $B_n^\vee$ $(n \geq 3)$.

(a) $\pm 2\varepsilon_i + 2r$ $(1 \leq i \leq n;\ r \in \mathbb{Z})$; $\pm\varepsilon_i \pm \varepsilon_j + r$ $(1 \leq i < j \leq n; r \in \mathbb{Z})$.

(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = 2\varepsilon_n.$

(c)

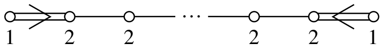


(1.3.4) *Type* C_n $(n \ge 2)$.

(a) $\pm 2\varepsilon_i + r\ (1 \le i \le n;\ r \in \mathbb{Z}); \quad \pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n; r \in \mathbb{Z}).$

(b) $a_0 = -2\varepsilon_1 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = 2\varepsilon_n.$

(c)

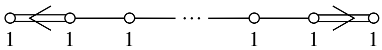


(1.3.5) *Type* $C_n^\vee$ $(n \ge 2)$.

(a) $\pm\varepsilon_i + \frac{1}{2}r\ (1 \le i \le n;\ r \in \mathbb{Z}); \quad \pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z}).$

(b) $a_0 = -\varepsilon_1 + \frac{1}{2}, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_n.$

(c)



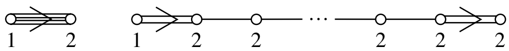
(1.3.6) *Type* BC_n $(n \ge 1)$.

(a) $\pm\varepsilon_i + r\ (1 \le i \le n;\ r \in \mathbb{Z}); \quad \pm 2\varepsilon_i + 2r + 1\ (1 \le i \le n;\ r \in \mathbb{Z});$

$$\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z}).$$

(b) $a_0 = -2\varepsilon_1 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_n.$

(c)



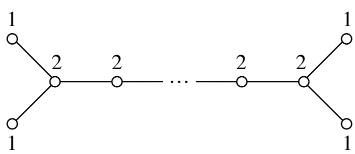
$(n = 1)$ $(n \geqslant 2)$

(1.3.7) *Type* D_n $(n \ge 4)$.

(a) $\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z})$

(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1, \quad a_i = \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1), \quad a_n = \varepsilon_{n-1} + \varepsilon_n.$

(c)



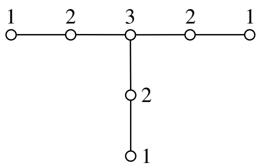
These are the "classical" reduced affine root systems. The next seven types ((1.3.8)–(1.3.14)) are the "exceptional" reduced affine root systems. In (1.3.8)–(1.3.10) let

$$\omega_i = \varepsilon_i - \frac{1}{9}(\varepsilon_1 + \cdots + \varepsilon_9) \qquad (1 \le i \le 9).$$

(1.3.8) *Type* E_6.

(a) $\pm(\omega_i - \omega_j) + r \ (1 \le i < j \le 6;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_j + \omega_k) + r \ (1 \le i < j < k \le 6;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_2 + \cdots + \omega_6) + r \ (r \in \mathbb{Z})$.

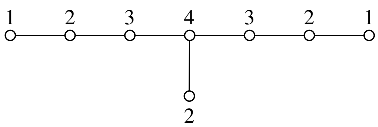
(b) $a_0 = -(\omega_1 + \cdots + \omega_6) + 1, \quad a_i = \omega_i - \omega_{i+1} \ (1 \le i \le 5)$,
$a_6 = \omega_4 + \omega_5 + \omega_6$.

(c) 

(1.3.9) *Type* E_7.

(a) $\pm(\omega_i - \omega_j) + r \ (1 \le i < j \le 7;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_j + \omega_k) + r \ (1 \le i < j < k \le 7;\ r \in \mathbb{Z})$;
$\pm(\omega_1 + \cdots + \hat{\omega}_i + \cdots + \omega_7) + r \ (1 \le i \le 7;\ r \in \mathbb{Z})$.

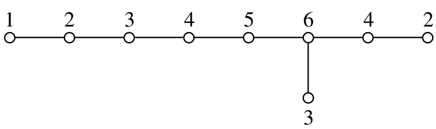
(b) $a_0 = -(\omega_1 + \cdots + \omega_6) + 1, \quad a_i = \omega_i - \omega_{i+1} \ (1 \le i \le 6)$,
$a_7 = \omega_5 + \omega_6 + \omega_7$.

(c) 

(1.3.10) *Type* E_8.

(a) $\pm(\omega_i - \omega_j) + r \ (1 \le i < j \le 9;\ r \in \mathbb{Z})$;
$\pm(\omega_i + \omega_j + \omega_k) + r \ (1 \le i < j < k \le 9;\ r \in \mathbb{Z})$.

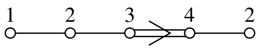
(b) $a_0 = \omega_1 - \omega_2 + 1, \quad a_i = \omega_{i+1} - \omega_{i+2} \ (1 \le i \le 7)$,
$a_8 = \omega_7 + \omega_8 + \omega_9$.

(c) 

(1.3.11) *Type* F_4.

(a) $\pm\varepsilon_i + r\ (1 \le i \le 4;\ r \in \mathbb{Z})$; $\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le 4;\ r \in \mathbb{Z})$;
$\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) + r\ (r \in \mathbb{Z})$.

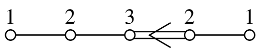
(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = \varepsilon_3 - \varepsilon_4$, $a_3 = \varepsilon_4$,
$a_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$.

(c) 

(1.3.12) *Type* $F_4^\vee$.

(a) $\pm 2\varepsilon_i + 2r\ (1 \le i \le 4;\ r \in \mathbb{Z})$; $\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le 4;\ r \in \mathbb{Z})$;
$\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 + 2r\ (r \in \mathbb{Z})$.

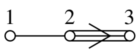
(b) $a_0 = -\varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = \varepsilon_3 - \varepsilon_4$, $a_3 = 2\varepsilon_4$,
$a_4 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4$.

(c) 

(1.3.13) *Type* G_2.

(a) $\pm(\varepsilon_i - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)) + r\ (1 \le i \le 3;\ r \in \mathbb{Z})$;
$\pm(\varepsilon_i - \varepsilon_j) + r\ (1 \le i < j \le 3;\ r \in \mathbb{Z})$.

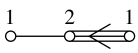
(b) $a_0 = \varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = \varepsilon_3 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$.

(c) 

(1.3.14) *Type* $G_2^\vee$.

(a) $\pm(3\varepsilon_i - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)) + 3r\ (1 \le i \le 3;\ r \in \mathbb{Z})$;
$\pm(\varepsilon_i - \varepsilon_j) + r\ (1 \le i < j \le 3;\ r \in \mathbb{Z})$.

(b) $a_0 = \varepsilon_1 - \varepsilon_2 + 1$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = 3\varepsilon_3 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$.

(c) 

We come now to the non-reduced affine root systems. In the Dynkin diagrams below, an asterisk placed over a vertex indicates that if a_i is the affine root corresponding to that vertex in a basis of S, then $2a_i \in S$.

(1.3.15) *Type* (BC_n, C_n) $(n \ge 1)$.

(a) $\pm\varepsilon_i + r$, $\pm 2\varepsilon_i + r\ (1 \le i \le n, r \in \mathbb{Z})$;
$\pm\varepsilon_i \pm \varepsilon_j + r\ (1 \le i < j \le n;\ r \in \mathbb{Z})$.