INTRODUCTION TO CIRCLE PACKING

The topic of "circle packing" was born of the computer age but takes its inspiration and themes from core areas of classical mathematics. A circle packing is a configuration of circles having a specified pattern of tangencies, as introduced by William Thurston in 1985. This book lays out their study, from first definitions to the latest theory, computations, and applications. The topic can be enjoyed for the visual appeal of the packing images – over 200 in the book – and the elegance of circle geometry, for the clean line of theory, for the deep connections to classical topics, or for the emerging applications. Circle packing has an experimental and visual character that is unique in pure mathematics, and the book exploits that character to carry the reader from the very beginnings to links with complex analysis and Riemann surfaces. There are intriguing, often very accessible, open problems throughout the book and seven Appendices on subtopics of independent interest. This book lays the foundation for a topic with wide appeal and a bright future.

INTRODUCTION TO CIRCLE PACKING

The Theory of Discrete Analytic Functions

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To my lovely wife, Dolores,
and our two wonderful children, Laura and Jon

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Preface

The circle is arguably the most studied object in all of mathematics, so it was a surprise at a conference in 1985 to hear William Thurston introducing a new topic called circle packing. And if encountering a new idea is one of the pleasures of mathematics, then seeing it attach to your favorite topic is a true joy. When Thurston conjectured a connection between circle packing and the venerable Riemann Mapping Theorem of 1851, I was hooked.

Now, nearly 20 years later, one sees that this was no mere passing encounter for these topics. Circle packing has opened a discrete world that both parallels and approximates the classical world of conformal geometry – a "quantum" complex analysis that is classical in the limit. In this book, I introduce circle packing as a portal into the beauties of conformal geometry, while I use the classical theory as a roadmap for developing circle packing.

Circle packings are configurations of circles with specified patterns of tangency. They should not be confused with sphere packings; here, it is the pattern of tangencies that is central – the connection between combinatorics and geometry. We will study the existence, uniqueness, computation, manipulation, display, and application of circle packings from the ground up. There are no formal prerequisites for this study; indeed, I shamelessly exploit the visual nature of circle packing and our native intuition about circles so that even the novice mathematician can penetrate deeply into the subject. At the same time, I have an obligation to circle packing itself as a new field, so the book is mathematically rigorous.

To balance access with rigor, I have structured the book in four parts. For most readers, Part I will be the first encounter with circle packings, so it is a broad overview: from a circle packing managerie to the function-theory paradigm. We become more formal in Part II with a proof of the fundamental existence and uniqueness result for maximal packings from first principles. Because the key roles are played by surprisingly elementary geometric arguments, this can serve even the non-mathematician as an exemplar of a robust and self-contained mathematical theory. In the classroom, Part II serves well as a one semester course for advanced undergraduate or beginning graduate students.

I define a discrete analytic function theory based on circle packings in Part III. At its core, analyticity is a profoundly geometric property, and this comes out in the discrete setting in a very compelling way. The amazing integrity of the analogies is confirmed in Part IV, when we prove that under refinement, the objects of the discrete theory converge to their classical counterparts. We prove Thurston's 1985 conjecture on the approximation of conformal maps (the Rodin/Sullivan Theorem) from first principles and then extend it broadly to other functions and to conformal structures. The circle packing methods described here have made the 150-year-old Riemann Mapping Theorem computable in many situations for the very first time. I demonstrate a number of applications, the most surprising of which involves "cortical brain mapping." Material in Parts III and IV could augment the traditional complex analysis sequence or serve as a basis for advanced topics courses.

The book also provides a wealth of material for individual or group projects from the undergraduate to the research level. I promote an intuitive and hands-on approach throughout, posing many open questions and experimental opportunities; see in particular, the several independent topics in the appendixes. I have provided "practica" on computational and software issues for those willing to get their hands dirty, and one can always download and run my software package, `CirclePack`. The book closes with a full circle packing bibliography.

People are drawn to mathematics for any number of reasons, from the clarity in elementary geometry, the challenge of richly layered theory and open questions, the discipline of computation, to the need for results in other areas. Read with an open mind and you can find all of these in circle packing – and I have not even mentioned the pure aesthetic pleasure of the pictures, which can sustain us all through the rough patches. I hope you enjoy circle packing.

I have many people to thank for their advice, encouragement, and patience over the years of this book's writing. Thanks go to my circle packing collaborators and friends, from whom I've learned so much: Dov Aharonov, Phil Bowers, Charles Collins, and Alan Beardon; also to my former students Tomasz Dubejko and Brock Williams, great circle packers both. Special thanks to Alan Beardon and Fred Gehring for their unfailing encouragement and sage advice over the years; to Oded Schramm, Jim Cannon, and Bill Floyd for many enjoyable and insightful conversations; and to William Thurston for the audacious notion of circle packing. To my many friends in classical function theory: I'm still one of you!

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