Part I

An Overview of Circle Packing

Circle packing has arrived so recently on the mathematical scene as to be totally new to many readers. Therefore, I am devoting Part I to an informal and largely visual tour of the topic, introducing only the most basic terminology and notation, but giving the reader a glimpse of how the full story will unfold. We all understand circles, but the reader may be surprised at how deeply they can carry us into the heart of conformal geometry.

Chapter 1 begins with a visit to a "menagerie" of circle packings, a wide-ranging collection that I hope you enjoy as much in the touring as I did in the collecting. The Menagerie suggests our first theme: Given a specified combinatoric pattern, what can one say about the existence, uniqueness, and variety of circle packings having those combinatorics? In Chapter 2 we get a view of the landscape beyond existence and uniqueness, where the central theme of the book – the emergence of fundamental parallels with analytic functions – plays out. I hope to set a style here that will carry on throughout the book, namely, one that engages the reader's native intuition not about static pictures, but about packing dynamics: How does a packing react if we change its boundary radii? If we introduce branching? How are the combinatorics and geometry feeding off one another? Deep classical themes will bubble to the surface with a surprising ease and clarity if one only remains open to the possibilities.

At the end of Part I is the first of four practica inviting the reader into the experimental side of the topic. Circle packings exist both in theory and in *fact*. The book requires nothing more than mental experiments, but those with an adventurous spirit may wish to grapple with my software package `CirclePack` or even do their own coding.

1
A Circle Packing Menagerie

1.1. First Views

In the belief that images speak louder than words, I will begin with a preliminary ramble through a menagerie of circle packings. Look for the common features and the differences in preparation for the guided tour to follow.

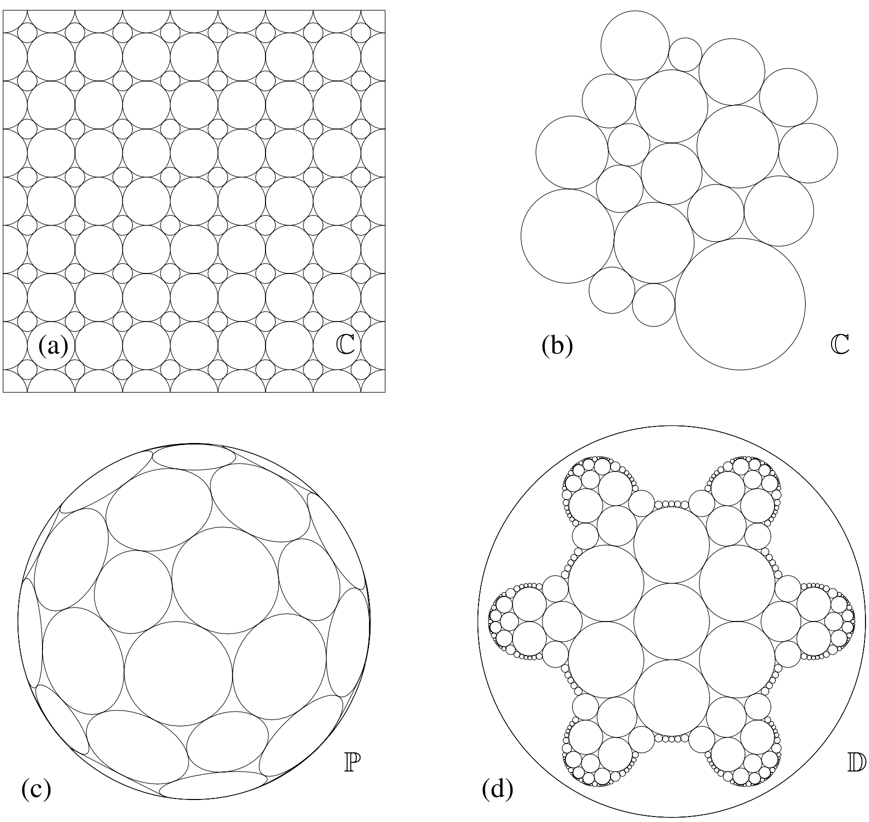

Figure 1.1. Collection 1.

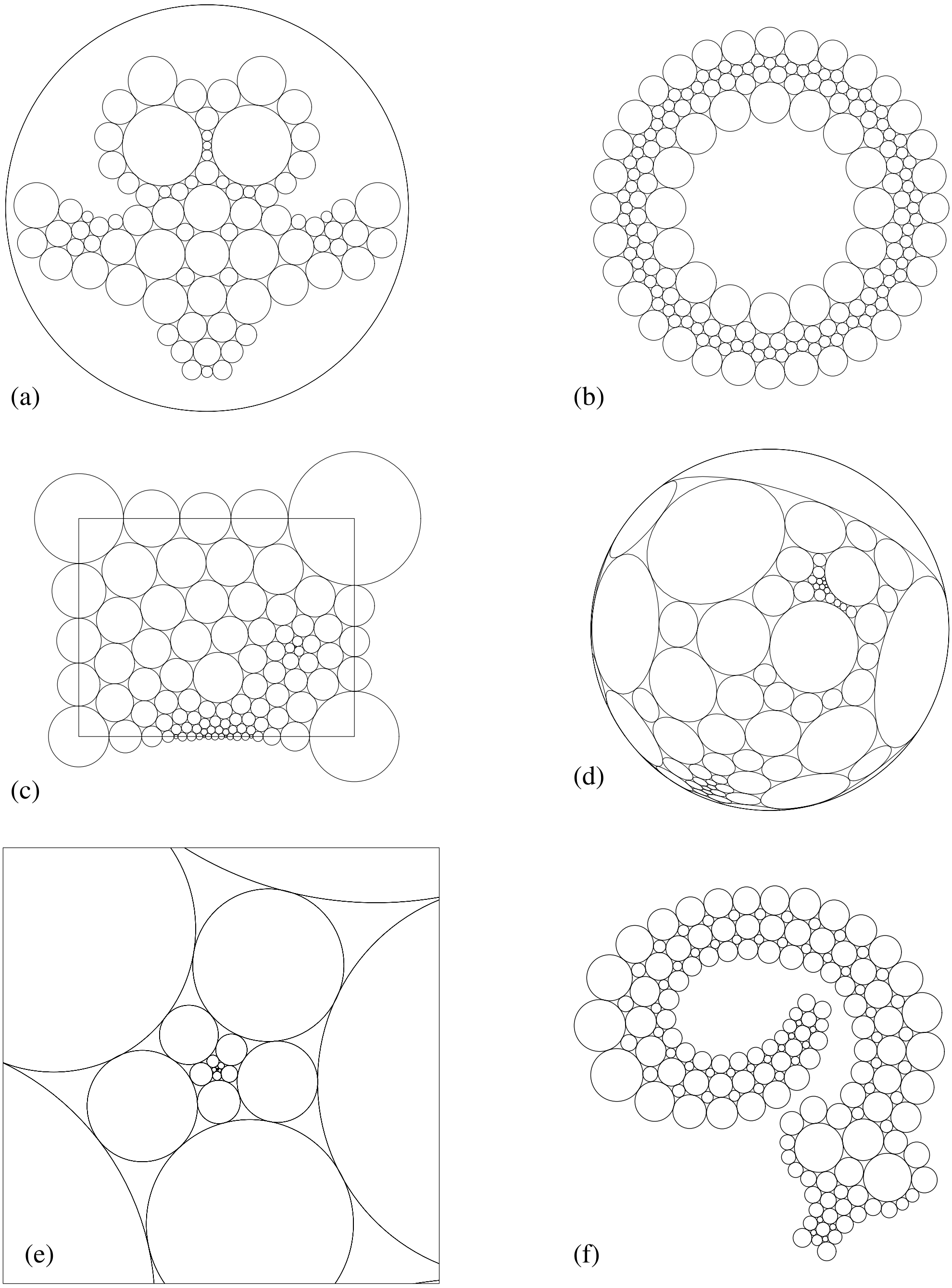

Figure 1.2. Collection 2.

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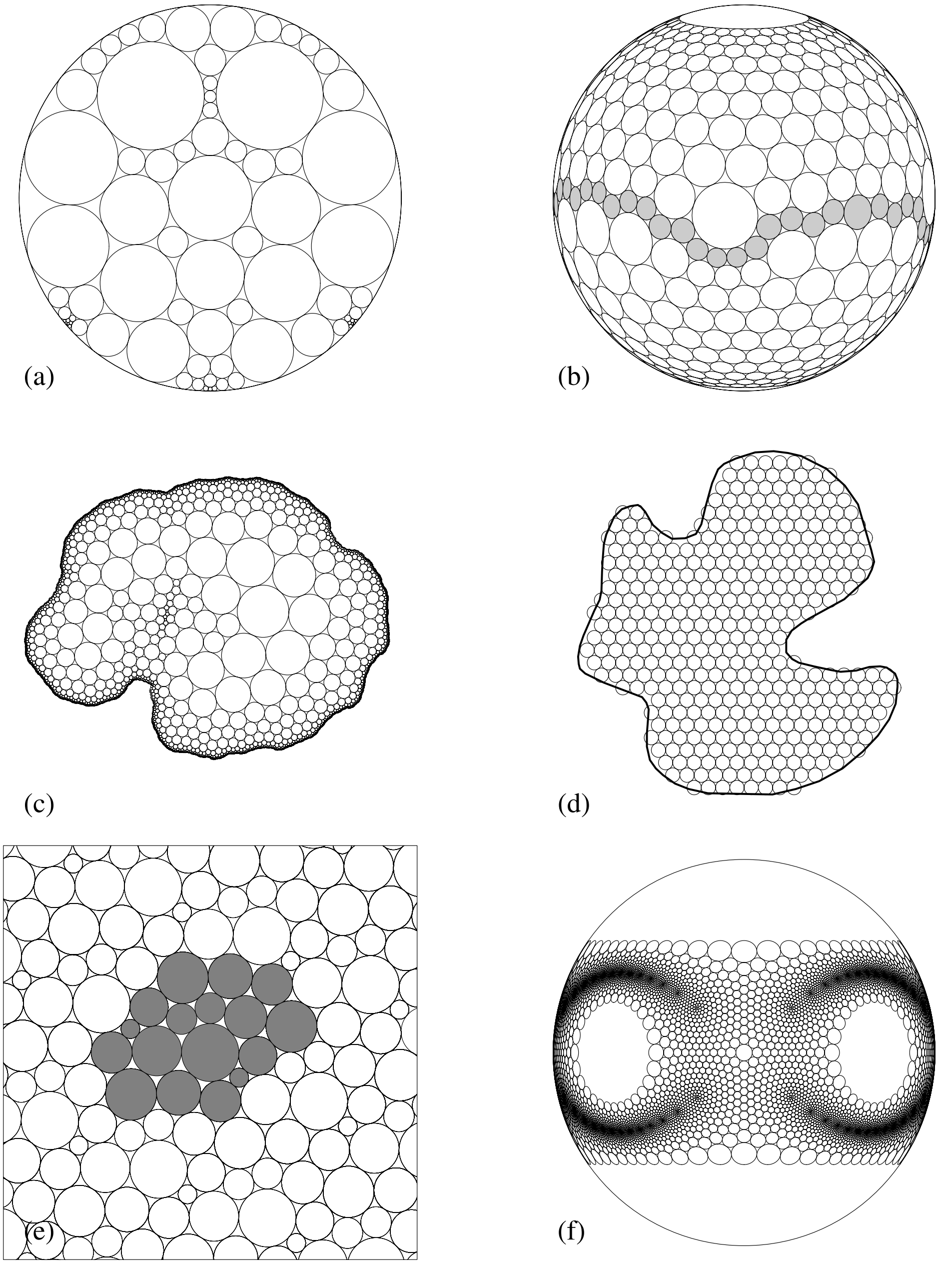

Figure 1.3. Collection 3.

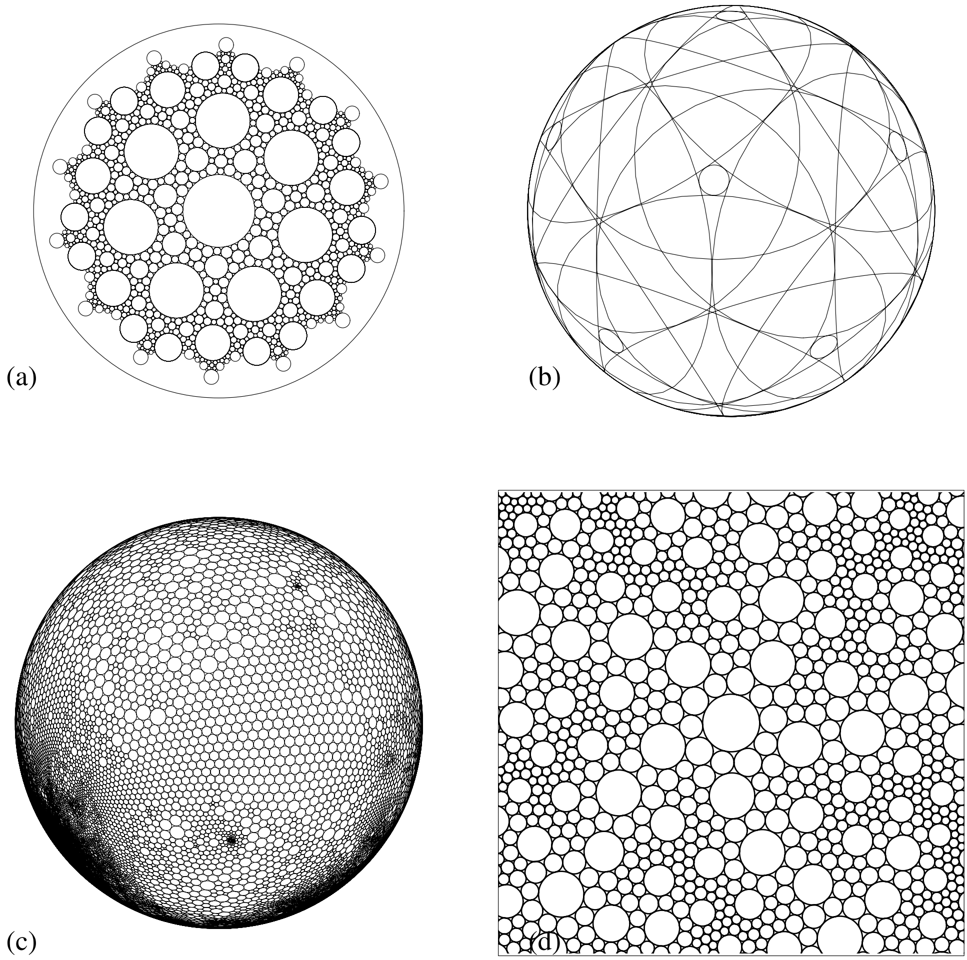

Figure 1.4. Collection 4.

The examples we have seen so far have been relatively tame. Before we start our guided tour, let us turn up the heat a notch with the more involved examples in Figure 1.4. Some of these reflect the complexity of applications; others contain internal symmetries, the more subtle of which may not be immediately evident. They represent topics we will touch upon in Parts III and IV of the book.

1.2. A Guided Tour

We start with some basics that you may already have deduced. First, there are three geometric settings for our packings, *euclidean*, *spherical*, and *hyperbolic*. In Figure 1.1

we see the familiar *euclidean plane* $\mathbb{R}^2$ in (a) and (b), the sphere in (c), and the interior of a disc in (d). Throughout the book, we treat $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and make use of complex arithmetic. The sphere will be the *Riemann sphere*, represented as the ordinary unit sphere in $\mathbb{R}^3$ and denoted by $\mathbb{P}$ (for *complex projective space*). The outer circle enclosing the packing of Figure 1.1(d) is not part of the packing; rather it is the boundary of the unit disc $\mathbb{D}$ in the plane. Here, however, $\mathbb{D}$ represents the Poincaré disc, a standard model of the *hyperbolic plane*. The geometries of $\mathbb{P}$, $\mathbb{C}$, and $\mathbb{D}$ will be of central importance in our work and we will have more to say about their distinct personalities shortly.

Next, observe that each packing involves an underlying pattern of tangencies. The hierarchy of structures is indicated in Fig. 1.5. All our tangencies are *external*, each circle lying outside the disc bounded by the other. In fact, however, tangencies do not occur in isolated pairs; rather the fundamental units of the patterns are mutually tangent triples of circles (*triples*, for short), with each triple forming a (curvilinear) triangular *interstice*. Triples are as important to the rigidity associated with circle packings as cross-bracing is to the rigidity of a bookcase. In turn, the triples of a pattern are linked together through shared pairs of circles to form the next level of structure, the *flower*, consisting of a central circle and some number of *petal* circles, the chain of successively tangent neighbors. The number of petals defines the *degree* of the central circle. The condition that every

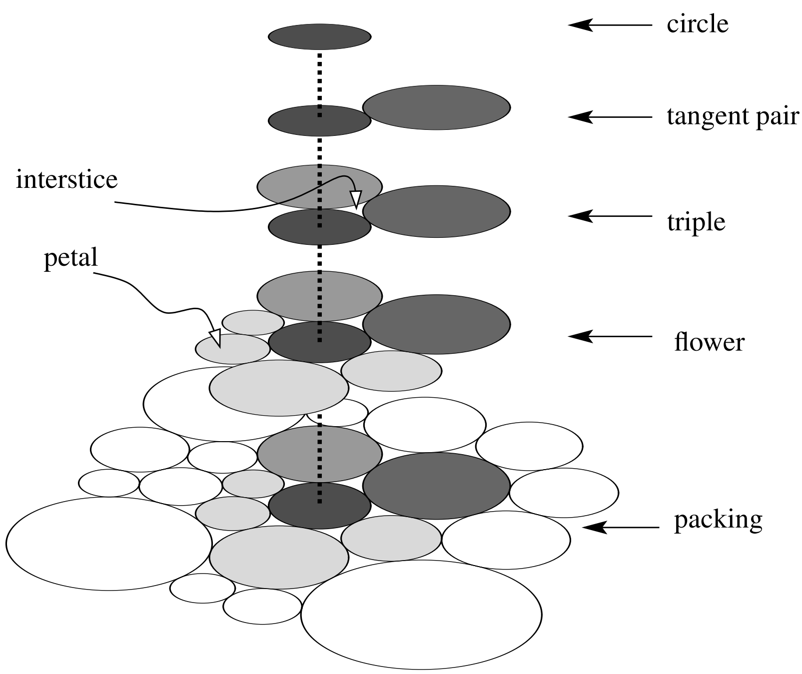

Figure 1.5. A hierarchy of circle packing structure.

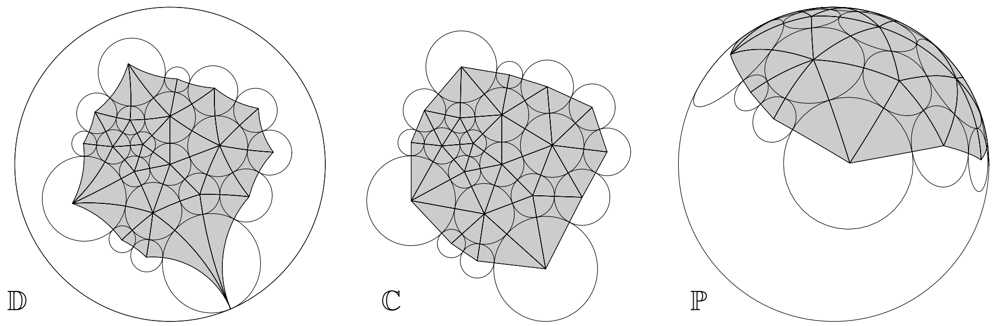

Figure 1.6. Carriers in the three geometries.

circle have such a flower is a *local planarity* condition that we will enforce on all our packings. Last, of course, the overall pattern of a packing consists of interlinked flowers. There will be some mild additional restrictions when we get to the function theory of Part III.

Within the context of the pattern of tangencies, further variability resides with the radii of the circles. Indeed, each triple may be thought of as a discrete piece of "geometry," its shape determined by the three radii involved. But in sewing these pieces together, even the first step, formation of a flower from several triples, involves a compatibility condition on the ensemble. And then, of course, the flowers must further cooperate to form the overall pattern. So although one might say that the combinatorics provide a master plan, it is the patient cooperation of the individual circles that allows that plan to be realized geometrically as a circle packing.

Various auxiliary structures in a circle packing may highlight, or even suggest, important properties. Foremost among the structures is the *carrier*, obtained by connecting the centers of tangent circles with geodesic segments to form *edges* and *faces*. The carrier is a geometric realization of a packing's abstract combinatorics. The packing of Figure 1.5 has been transplanted to Figure 1.6, where it is shown with its carrier in the hyperbolic, euclidean, and spherical settings.

Identifying various portions of a packing or carrier, appropriately grouping faces or circles, even coloring for visual effect, can make all the difference in understanding a particular packing. I have reproduced the packings of Figure 1.4 in Figure 1.7 with various decorations. The shaded faces in Figure 1.7(a) reproduce one of the most famous illustrations from 19th-century function theory, a fundamental domain for the *Klein surface*. The packing in Figure 1.7(b) is a *branched* packing and the dark circles are among the 12 of degree 5 at which the branching occurs. (Incidentally, this packing has the same combinatorics as Figure 1.1(c).) The packing in Figure 1.7(c) represents a flattened cortical surface from a magnetic resonance imaging (MRI) scan of a human cerebellum; colors are essential in visually distinguishing various cortical lobes. By drawing only certain edges in Figure 1.7(d) we reveal a hidden tiling, and even more selective darkened edges show an emerging fractal curve. We will use decorations like these to highlight features of packings throughout the book.

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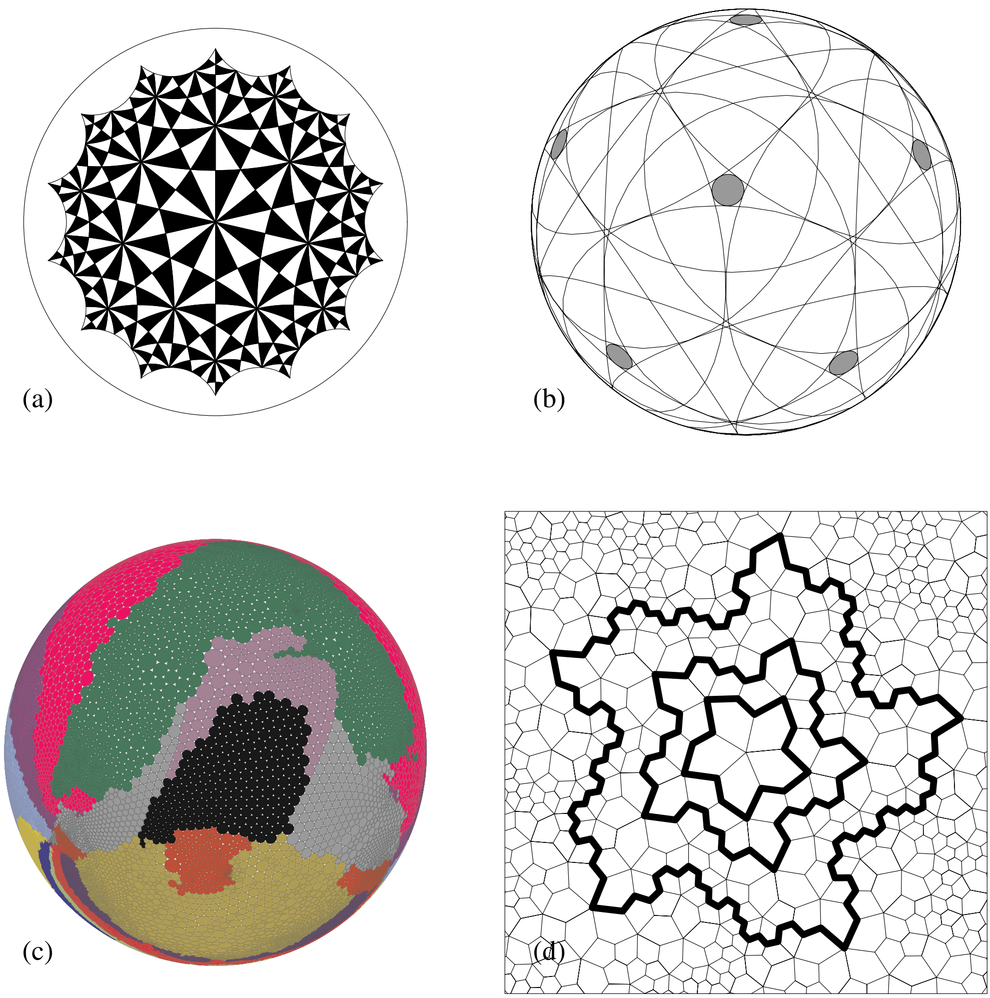

Figure 1.7. Decorations often make the packing.

Putting aside these decorations, there are two central sources of flexibility in circle packing: *combinatoric* and *geometric*. The clash and cooperation between these two provides the speciation seen in our Menagerie. Let us pin down some of that variability.

1.2.1. Combinatorial Flexibility

Combinatorial flexibility has to do with the pattern of tangencies. Circle packings may be finite or infinite; some circles will be *interior* to their patterns, fully surrounded by their flowers, while others are *boundary* circles, ones at the edge with flowers that do not close up. Combinatorics may be conceived abstractly, derived from some natural configuration, as with lattices, or obtained from existing packings. Thus Figure 1.3(d) was

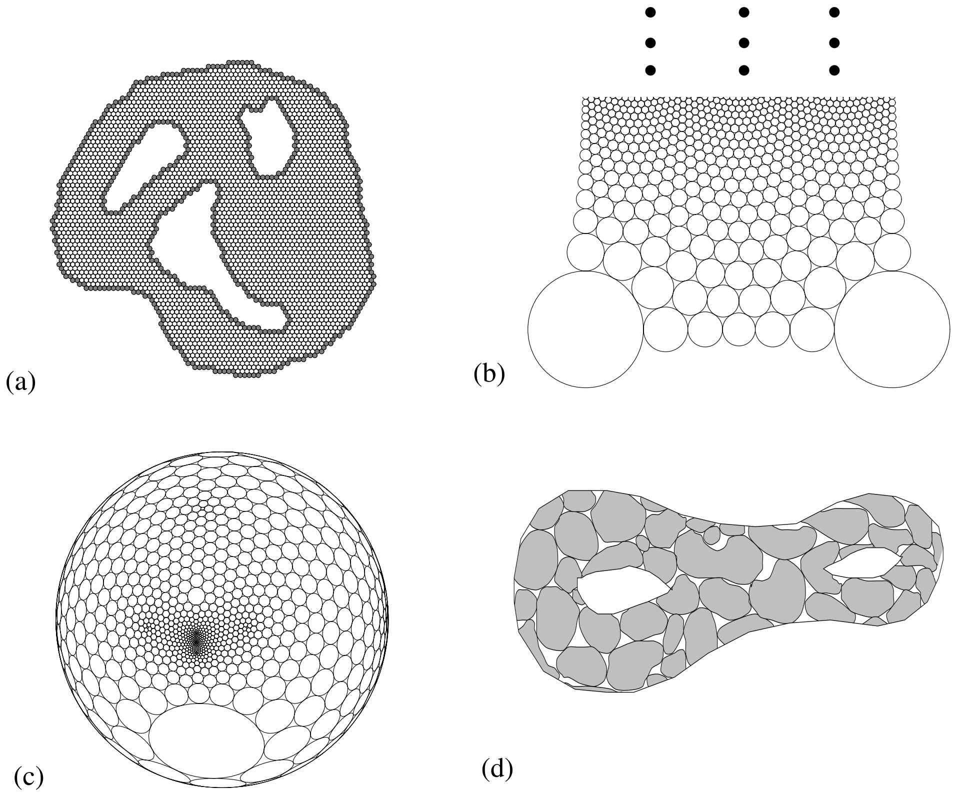

Figure 1.8. Illustrations of topological variety.

cookie-cut from an infinite packing, while Figure 1.3(b) resulted from an abstract gluing of two patterns, the line of circles showing the weld. Topological variety is illustrated in Figure 1.8. Packings can have more than one boundary component and even infinite packings can have boundary circles. Finite packings might have no boundary, as in the packing of the sphere in (c) or the packing of a compact genus 2 surface suggested by (d). (Note with regard to the latter that we will need to clarify the sense in which these are "circles" – what metric is in play?)

Certain of our patterns have highly regular combinatorics. Archetypes are the constant-degree complexes $K^{[n]}$ in which every circle has n neighbors. Figure 1.9 shows us a packing of $K^{[5]}$ in the sphere and a packing of $K^{[7]}$ in the disc. And everyone will recognize in the background the familiar "penny" packing, the *regular hexagonal packing* of the plane for $K^{[6]}$ (also denoted H). Each of these packings has circles of constant radius in its respective geometry. Since we are talking combinatorics, however, you might note that Figure 1.2(e) is also hexagonal, though quite distinct globally from a penny packing.

The *ball-bearing* motif of Figure 1.1(a) has a nice rectangular feel, but of course the large circles alone would leave quadrangular *versus* triangular interstices, so we have to add in the smaller ball-bearing circles to meet our basic triples requirement.

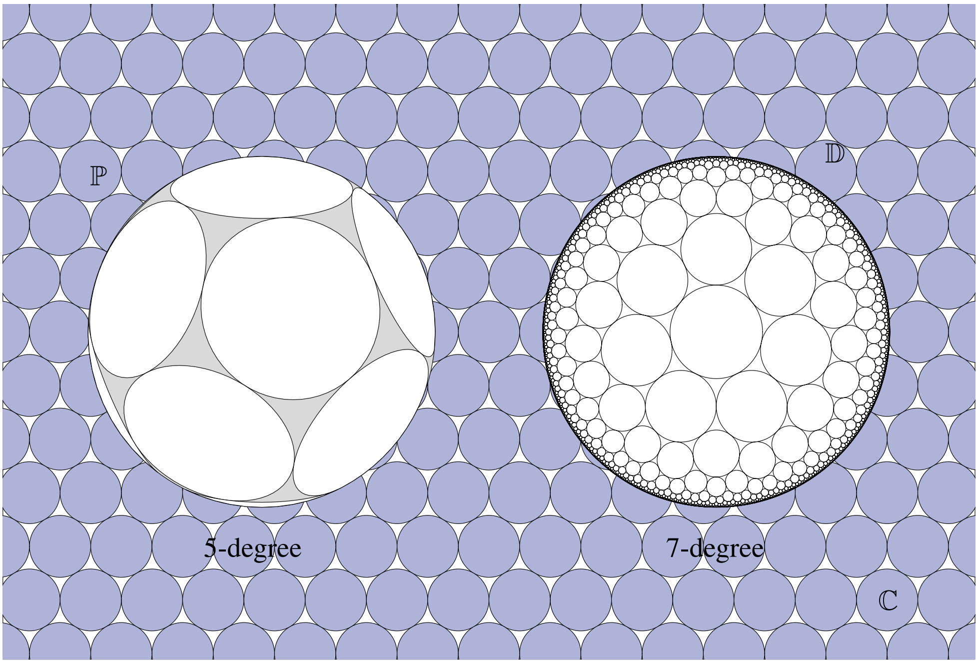

Figure 1.9. Constant degree packings: 5-, 6-, and 7-degree.

Combinatorial symmetries will be common in our work, but they can be very subtle. In Figure 1.3(e), for instance, one can with careful scrutiny and the aid of the shaded circles confirm that there is an underlying double translational symmetry; we will see later that this shaded fragment should be treated as a circle packing lying on a torus.

Several examples, such as Figure 1.3(c), were chosen to emphasize that no combinatorial regularity is required a priori. In point of fact, there are few combinatoric restrictions on our circle packings; beyond the requirement that the circles form triples and flowers, we have natural connectivity and orientability conditions – all will be described in detail at the appropriate time. We ultimately require only *sensible* combinatorics.

Keep in mind through all these examples that the pattern of tangencies of a packing is a combinatoric and not a geometric notion. It is the radii which imbue the pattern with size and shape, so let us move to the discussion of that source of variability.

1.2.2. Geometric Flexibility

Check out Figures 1.2(a) and 1.3(a). See any similarity? They're clearly distinct packings, but when they are shown side by side in Figure 1.10, one can – with a magnifying class and great patience – verify that they have the same underlying combinatorics. One could number the circles of P, say, and then transfer that numbering to Q in such a way that