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# 160 Approximation by Algebraic Numbers

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# Preface

Und alles ist mir dann immer wieder zerfallen, auf dem Konzentrationshöhepunkt ist mir dann immer wieder alles zerfallen.

*Thomas Bernhard*

(...) il faut continuer, je vais continuer.

*Samuel Beckett*

The central question in Diophantine approximation is: how well can a given real number $\xi$ be approximated by rational numbers, that is, how small can the difference $|\xi - p/q|$ be made for varying rational numbers $p/q$? The accuracy of the approximation of $\xi$ by $p/q$ is being compared with the 'complexity' of the rational number $p/q$, which is measured by the size of its denominator $q$. It follows from the theory of continued fractions (or from Dirichlet's Theorem) that for any irrational number $\xi$ there exist infinitely many rational numbers $p/q$ with $|\xi - p/q| < q^{-2}$. This can be viewed as the first general result in this area.

There are two natural generalizations of the central question. On the one hand, one can treat rational numbers as algebraic numbers of degree one and study, for a given positive integer $n$, how well $\xi$ can be approximated by algebraic numbers of degree at most $n$. On the other hand, $\xi - p/q$ can be viewed as $q\xi - p$, that is as $P(\xi)$, where $P(X)$ denotes the integer polynomial $qX - p$. Thus, for a given positive integer $n$, one may ask how small $|P(\xi)|$ can be made for varying integer polynomials $P(X)$ of degree at most $n$. To do this properly, one needs to define a notion of size, or 'complexity', for algebraic numbers $\alpha$ and for integer polynomials $P(X)$, and we have to compare the accuracy of the approximation of $\xi$ by $\alpha$ (*resp.* the smallness of $|P(\xi)|$) with the size of $\alpha$ (*resp.* of $P(X)$). In both cases, we use for the size the naive height H: the height $\mathrm{H}(P)$ of $P(X)$ is the maximum of the absolute values of its coefficients and the height $\mathrm{H}(\alpha)$ of $\alpha$ is that of its minimal polynomial over $\mathbb{Z}$.

In 1932, Mahler proposed to classify the real numbers (actually, the complex numbers) into several classes according to the various types of answers to the latter question, while in 1939 Koksma introduced an analogous classification based on the former question. In both cases, the algebraic numbers form one of the classes. Let $\xi$ be a real number and let $n$ be a positive integer. According to Mahler, we denote by $w_n(\xi)$ the supremum of the real numbers $w$ for which there exist infinitely many integer polynomials $P(X)$ of degree at most $n$ satisfying

$$0 < |P(\xi)| \le \mathrm{H}(P)^{-w},$$

and we divide the set of real numbers into four classes according to the behaviour of the sequence $(w_n(\xi))_{n\ge1}$. Following Koksma, we denote by $w_n^*(\xi)$ the supremum of the real numbers $w$ for which there exist infinitely many real algebraic numbers $\alpha$ of degree at most $n$ satisfying

$$0 < |\xi - \alpha| \le \mathrm{H}(\alpha)^{-w-1}.$$

It turns out that both classifications coincide, inasmuch as each of the four classes defined by Mahler corresponds to one of the four classes defined by Koksma. However, they are not strictly equivalent, since there exist real numbers $\xi$ for which $w_n(\xi)$ and $w_n^*(\xi)$ differ for any integer $n$ at least equal to 2. In addition, it is a very difficult (and, often, still open) question to determine to which class a given ‘classical’ number like $\pi$, $e$, $\zeta(3)$, $\log 2$, etc. belongs.

The present book is mainly concerned with the following problem: given two non-decreasing sequences of real numbers $(w_n)_{n\ge1}$ and $(w_n^*)_{n\ge1}$ satisfying some necessary restrictions (e.g. $w_n^* \le w_n \le w_n^* + n - 1$), does there exist a real number $\xi$ with $w_n(\xi) = w_n$ and $w_n^*(\xi) = w_n^*$ for all positive integers $n$? This question is very far from being solved, although we know (see Chapter 4) that almost all (in the sense of the Lebesgue measure on the line) real numbers share the same approximation properties, namely they satisfy $w_n(\xi) = w_n^*(\xi) = n$ for all positive integers $n$.

There are essentially two different points of view for investigating such a problem. We may try to construct explicitly (or semi-explicitly) real numbers with the required properties (Chapter 7) or, if this happens to be too difficult, we may try to prove the existence of real numbers with a given property by showing that the set of these numbers has positive Hausdorff dimension

(Chapters 5 and 6). Most often, however, we are unable to exhibit a single explicit example of such a number.

The content of the present book is as follows. Chapter 1 is devoted to the approximation by rational numbers. We introduce the notion of continued fractions and establish their main properties needed to prove the celebrated metric theorem of Khintchine saying that, for a continuous function $\Psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ such that $x \mapsto x\Psi(x)$ is non-increasing, the equation $|q\xi - p| < \Psi(q)$ has infinitely many integer solutions $(p, q)$ with $q$ positive for either almost no or almost all real numbers $\xi$, according to whether the sum $\sum_{q=1}^{+\infty} \Psi(q)$ converges or diverges.

In Chapter 2, we briefly survey the approximation to algebraic numbers by algebraic numbers, and recall many classical results (including Roth's Theorem and Schmidt's Theorem). We make a clear distinction between effective and ineffective statements.

Mahler's and Koksma's classifications of real numbers are defined in Chapter 3, where we show, following ideas of Wirsing, how closely they are related. Some links between simultaneous rational approximation and these classifications are also mentioned, and we introduce four other functions closely related to $w_n$ and $w_n^*$.

In Chapter 4, we establish Mahler's Conjecture to the effect that almost all real numbers $\xi$ satisfy $w_n(\xi) = w_n^*(\xi) = n$ for all positive integers $n$. This result, first proved by Sprindžuk in 1965, has been refined and extended since that time and we state the most recent developments, essentially due to a new approach found by Kleinbock and Margulis.

Exceptional sets are investigated from a metric point of view in Chapters 5 and 6. To this end, we introduce a classical powerful tool for discriminating between sets of Lebesgue measure zero, namely the notion of Hausdorff dimension. We recall the basic definitions and some well-known results useful in our context. This allows us to prove the theorem of Jarník and Besicovitch saying that for any real number $\tau \geq 1$ the Hausdorff dimension of the set of real numbers $\xi$ with $w_1^*(\xi) \geq 2\tau - 1$ is equal to $1/\tau$. We also establish its generalization to any degree $n$ (with $w_1^*(\xi) \geq 2\tau - 1$ replaced by $w_n^*(\xi) \geq (n+1)\tau - 1$) obtained in 1970 by A. Baker and Schmidt. Chapter 6 is devoted to refined statements and contains general metric theorems on sets of real numbers which are very close to infinitely many elements of a fixed set of points which are, in some sense, evenly distributed.

In Chapter 7, we prove, following ideas of Schmidt, that the class formed by the real numbers $\xi$ with $\limsup_{n\to+\infty} w_n(\xi)/n$ infinite and $w_n(\xi)$ finite for all positive integers $n$ is not empty. At the same time, we show that there exist real numbers $\xi$ for which the quantities $w_n(\xi)$ and $w_n^*(\xi)$ differ by preassigned values, a result due to R. C. Baker. The real numbers $\xi$ with the required property are obtained as limits of sequences of algebraic numbers. This illustrates the importance of results on approximation of algebraic numbers by algebraic numbers. The remaining part of Chapter 7 is concerned with some other (simpler) explicit constructions.

Mahler's and Koksma's classifications emphasize the approximation by algebraic numbers of bounded degree. We may as well exchange the roles played by degree and height or let both vary simultaneously. We tackle this question in Chapter 8 by considering the classification introduced by Sprindžuk in 1962 and the so-called 'order functions' defined by Mahler in 1971. Further, some recent results of Laurent, Roy, and Waldschmidt expressed in terms of a more involved notion of height (namely, the absolute logarithmic height) are given.

In Chapter 9, we briefly discuss approximation in the complex field, in the Gaussian field, in $p$-adic fields, and in fields of formal Laurent series.

Chapter 10, which begins by a brief survey on the celebrated Littlewood Conjecture, offers a list of open questions. We hope that these will motivate further research.

Finally, there are two appendices. Appendix A is devoted to lemmas on zeros of polynomials: all proofs are given in detail and the statements are the best known at the present time. Appendix B lists classical auxiliary results from the geometry of numbers.

The Chapters are largely independent of each other.

We deliberately do not give proofs to all the theorems quoted in the main part of the text. We have clearly indicated when this is the case (see below). Furthermore, we try, in the end-of-chapter notes, to be as exhaustive as possible and to quote less-known papers, which, although interesting, did not yield up to now to further research. Of course, exhaustivity is an impossible task, and it is clear that the choice of the references concerning works at the border of the main topic of this book reflects the personal taste and the limits of the knowledge of the author.

The purpose of the exercises is primarily to give complementary results, thus they are often an adaptation of an original research work to which the reader is directed.

There exist already many textbooks dealing, in part, with the subject of the present one, e.g., by Schneider [517], Sprindžuk [539, 540], A. Baker [44], Schmidt [510, 512], Bernik and Melnichuk [90], Harman [273], and Bernik and Dodson [86]. However, the intersection rarely exceeds two or three chapters. Special mention should be made to the wonderful book of Koksma [332], which contains an impressive list of references which appeared before 1936.

Maurice Mignotte and Michel Waldschmidt encouraged me constantly. Many colleagues sent me comments, remarks, and suggestions. I am very grateful to all of them. Special thanks are due to Guy Barat and Damien Roy, who carefully read several parts of this book.

The present book will be regularly updated on my institutional Web page:
`http://www-irma.u-strasbg.fr/~bugeaud/Book`

The following statements are not proved in the present book:

Theorems 1.13 to 1.15, 1.17, 1.20, 2.1 to 2.8, 3.7, 3.8, 3.10, 3.11, 4.4 to 4.7, Proposition 5.1, Theorems 5.7, 5.9, 5.10, Proposition 6.1, Theorems 6.3 to 6.5, 8.1, 8.5, 8.8 (partially proved), 9.1 to 9.8, Lemma 10.1, Theorems 10.1, B.3, and B.4.

The following statements are left as exercises:

Theorems 1.16, 1.18, 1.19, 5.4, 5.6, 6.2, 6.9, 6.10, 7.2, 7.3, 7.6, 8.4, 8.6, 8.12, and Proposition 8.1.

# Frequently used notation

deg degree.
positive strictly positive.
$\mathcal{N}$ infinite set of integers.
$[\cdot]$ integer part.
$\{\cdot\}$ fractional part.
$\|\cdot\|$ distance to the nearest integer.
An empty sum is equal to 0 and an empty product is equal to 1.
$\phi$ the Euler totient function.
$\log_i$ the $i$-fold iterated logarithm.
$\ll, \gg$ means that there is an implied constant.
Card the cardinality (of a finite set).
$c, c_i, \kappa$ constants.
$c(\mathrm{var}_1, \ldots, \mathrm{var}_m)$ constant depending (at most) on the variables $\mathrm{var}_1, \ldots, \mathrm{var}_m$.
$\mathbb{A}_n$ the set of real algebraic numbers of degree at most $n$.
H naive height, Ch. 2.
$\Lambda$ size, Ch. 8.
h absolute height, Ch. 8.
$M$ Mahler's measure, App. A.
$\lambda$ the Lebesgue measure on the real line.
$\lambda(I) = |I|$ the Lebesgue measure of an interval $I$.
vol the $n$-dimensional Lebesgue measure.
$\mu$ a measure (not the Lebesgue one).
$\xi$ the number we approximate.
$d$ the degree of $\xi$ (when $\xi$ is algebraic).
$\alpha$ the algebraic approximant of $\xi$.
$n$ (an upper bound for) the degree of the approximant $\alpha$.
$H$ (an upper bound for) the height of the approximant $\alpha$.

$\underline{x}$ the $n$-tuple $(x_1, \ldots, x_n)$.
$\mathcal{B}$ set of badly approximable real numbers, Ch. 1.
$\mathcal{B}_M$ set of real numbers whose partial quotients are at most equal to $M$, Ch. 5.
$\Psi$, $x \mapsto x^\tau$ approximation functions, Ch. 1, 5, 6.
$\mathcal{K}_n^*(\Psi)$, $\mathcal{K}_n(\tau)$, $\mathcal{K}_n^*(\tau)$, $\mathcal{K}_n^+(\tau)$, $\mathcal{K}_{\mathcal{S}}^*(\Psi)$ Ch. 1, 5, 6.
$w_n(\xi, H)$, $w_n^*(\xi, H)$, $w_n^{*c}(\xi, H)$, $w_n(\xi)$, $w_n^*(\xi)$, $w_n^{*c}(\xi)$, $w(\xi)$, $w^*(\xi)$ Ch. 3
$w_n'(\xi)$, $\hat{w}_n'(\xi)$, $\hat{w}_n(\xi)$, $\hat{w}_n^*(\xi)$ Ch. 3
$A$-, $S$-, $T$-, $U$-, $A^*$-, $S^*$-, $T^*$-, $U^*$-numbers Ch. 3.
$U_m$-numbers Ch. 7.
$t(\xi)$, $t^*(\xi)$ the type and the $*$-type of $\xi$, Ch. 3.
$\Pi_+$ Ch. 4.
$w_n^+(\xi)$ Ch. 5.
$\mathcal{W}_n(\tau)$, $\mathcal{W}_n^*(\tau)$, $\mathcal{W}_n^+(\tau)$, $\mathcal{W}_n^{\geq}(\tau)$ Ch. 5.
dim Hausdorff dimension, Ch. 5.
$\mathcal{H}^f$ Hausdorff $\mathcal{H}^f$-measure, Ch. 5.
$\mathcal{H}^s$ Hausdorff $s$-measure, Ch. 5.
$\prec$ order between dimension functions, Ch. 5.
$\Phi$, $\Xi$ increasing functions, Ch. 5.
$\mathcal{S}$ infinite set of real numbers, Ch. 5, 6.
$\lambda$ lower order at infinity, Ch. 6.
$\mathcal{M}_\infty^f$ net measure, Ch. 6.
$\tilde{w}(\xi, H)$, $\tilde{w}(\xi)$, $\tilde{\mu}(\xi, H)$, $\tilde{\mu}(\xi)$, $\tilde{w}^*(\xi, H)$, $\tilde{w}^*(\xi)$ Ch. 8.
$\tilde{A}$-, $\tilde{S}$-, $\tilde{T}$-, $\tilde{U}$-numbers Ch. 8.
$O(u \mid \xi)$, $O^*(u \mid \xi)$ order functions, Ch. 8.
$\asymp$ order between order functions, Ch. 8.
$\tau(\xi)$ transcendence type of $\xi$, Ch. 8.
Res the resultant of two polynomials, App. A.
Disc the discriminant of an integer polynomial, App. A.
$\mathcal{C}$ a bounded convex body, App. B.
$\lambda_1, \ldots, \lambda_n$ the successive minima of a bounded convex body, App. B.