1
Introduction

The aim of this book is to introduce the reader to classical integrable systems. Because the subject has been developed by several schools having different perspectives, it may appear fragmented at first sight. We develop here the thesis that it has a profound unity and that the various approaches are simply changes of point of view on the same underlying reality. The more one understands each approach, the more one sees their unity. At the end one gets a very small set of interconnected methods.

This fundamental fact sets the tone of the book. We hope in this way to convey to the reader the extraordinary beauty of the structures emerging in this field, which have illuminated many other branches of theoretical physics.

The field of integrable systems is born together with Classical Mechanics, with a quest for exact solutions to Newton's equations of motion. It turned out that apart from the Kepler problem which was solved by Newton himself, after two centuries of hard investigations, only a handful of other cases were found. In the nineteenth century, Liouville finally provided a general framework characterizing the cases where the equations of motion are "solvable by quadratures". All examples previously found indeed pertained to this setting. The subject stayed dormant until the second half of the twentieth century when Gardner, Greene, Kruskal and Miura invented the Classical Inverse Scattering Method for the Korteweg–de Vries equation, which had been introduced in fluid mechanics. Soon afterwards, the Lax formulation was discovered, and the connection with integrability was unveiled by Faddeev, Zakharov and Gardner. This was the signal for a revival of the domain leading to an enormous amount of results, and truly general structures emerged which organized the subject. More recently, the extension of these results to Quantum Mechanics already led to remarkable results and is still a very active field of research.

Let us give a general overview of the ideas we present in this book. They all find their roots in the notion of Lax pairs. It consists of presenting the equations of motion of the system in the form $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$, where the matrices $L(\lambda)$ and $M(\lambda)$ depend on the dynamical variables and on a parameter λ called the spectral parameter, and [,] denotes the commutator of matrices. The importance of Lax pairs stems from the following simple remark: the Lax equation is an isospectral evolution equation for the Lax matrix $L(\lambda)$. It follows that the curve defined by the equation $\det(L(\lambda) - \mu I) = 0$ is time-independent. This curve, called the spectral curve, can be seen as a Riemann surface. Its moduli contain the conserved quantities. This immediately introduces the two main structures into the theory: groups enter through the Lie algebra involved in the commutator $[M, L]$, while complex analysis enters through the spectral curve.

As integrable systems are rather rare, one naturally expects strong constraints on the matrices $L(\lambda)$ and $M(\lambda)$. Constructing consistent Lax matrices may be achieved by appealing to factorization problems in appropriate groups. Taking into account the spectral parameter promotes this group to a loop group. The factorization problem may then be viewed as a Riemann–Hilbert problem, a central tool of this subject.

In the group theoretical setting, solving the equations of motion amounts to solving the factorization problem. In the analytical setting, solutions are obtained by considering the eigenvectors of the Lax matrix. At any point of the spectral curve there exists an eigenvector of $L(\lambda)$ with eigenvalue μ. This defines an analytic line bundle $\mathcal{L}$ on the spectral curve with prescribed Chern class. The time evolution is described as follows: if $\mathcal{L}(t)$ is the line bundle at time t then $\mathcal{L}(t)\mathcal{L}^{-1}(0)$ is of Chern class 0, i.e. is a point on the Jacobian of the spectral curve. It is a beautiful result that this point evolves linearly on the Jacobian. As a consequence, one can express the dynamical variables in terms of theta-functions defined on the Jacobian of the spectral curve. The two methods are related as follows: the factorization problem in the loop group defines transition functions for the line bundle $\mathcal{L}$.

The framework can be generalized by replacing the Lax matrix by the first order differential equation $(\partial_\lambda - M_\lambda(\lambda))\Psi = 0$, where $M_\lambda(\lambda)$ depends rationally on λ. The solution Ψ acquires non-trivial monodromy when λ describes a loop around a pole of M_λ. The isomonodromy problem consists of finding all M_λ with prescribed monodromy data. The solutions depend, in general, on a number of continuous parameters. The deformation equations with respect to these parameters form an integrable system. The theta-functions of the isospectral approach are then promoted to more general objects called the tau-functions.

One can study the behaviour around each singularity of the differential operator quite independently. In the group theoretical version, the above extension of the framework corresponds to centrally extending the loop groups. Around a singularity the most general extended group is the group $GL(\infty)$ which corresponds to the KP hierarchy. It can be represented in a fermionic Fock space. Fermionic monomials acting on the vacuum yield decomposed vectors, which describe an infinite Grassmannian introduced by Sato. In this setting, the time flows are induced by the action of commuting one-parameter subgroups, and the tau-function is defined on the Grassmannian, i.e. the orbit of the vacuum, and characterizes it. Finally the Plücker equations of the Grassmannian are identified with the equations of motion, written in the bilinear Hirota form.

We have tried, as much as possible, to make the book self-contained, and to achieve that each chapter can be studied quite independently. Generally, we first explain methods and then show how they can be applied to particular examples, even though this does not correspond to the historical development of the subject.

In Chapter 2 we introduce the classical definition of integrable systems through the Liouville theorem. We present the Lax pair formulation, and describe the symplectic structure which is encoded into the so-called r-matrix form. In Chapter 3 we explain how to construct Lax pairs with spectral parameter, for finite and infinite-dimensional systems. The Lax matrix may be viewed as an element of a coadjoint orbit of a loop group. This introduces immediately a natural symplectic structure and a factorization problem in the loop group. We also introduce, at this early stage, the notion of tau-functions. In Chapter 4 we discuss the abstract group theoretical formulation of the theory. We then describe the analytical aspects of the theory in Chapter 5. In this setting, the action variables are g moduli of the spectral curve, a Riemann surface of genus g, and the angle variables are g points on it. We illustrate the general constructions by the examples of the closed Toda chain in Chapter 6, and the Calogero model in Chapter 7.

The following two Chapters, 8 and 9, describe respectively the isomonodromic deformation problem and the infinite Grassmannian. Soliton solutions are obtained using vertex operators. Chapters 10 and 11 are devoted to the classical study of the KP and KdV hierarchies. We develop and use the formalism of pseudo-differential operators which allows us to give simple proofs of the main formal properties. Finite-zone solutions of KdV allow us to make contact with integrable systems of finite dimensionality and soliton solutions.

In the next Chapter, 12, we study the class of Toda and sine-Gordon field theories. We use this opportunity to exhibit the relations between their conformal and integrable properties. The sine-Gordon model is presented in the framework of the Classical Inverse Scattering Method in Chapter 13. This very ingenious method is exploited to solve the sine-Gordon equation.

The last three chapters may be viewed as mathematical appendices, provided to help the reader. First we present the basic facts of symplectic geometry, which is the natural language to speak about Classical Mechanics and integrable systems. Since mathematical tools from Riemann surfaces and Lie groups are used almost everywhere, we have written two chapters presenting them in a concise way. We hope that they will be useful at least as an introduction and to fix notations.

Let us say briefly how we have limited our discussion. First we choose to remain consistently at a relatively elementary mathematical level, and have been obliged to exclude some important developments which require more advanced mathematics. We put the emphasis on methods and we have not tried to make an exhaustive list of integrable systems. Another aspect of the theory we have touched only very briefly, through the Whitham equations, is the study of perturbations of integrable systems. All these subjects are very interesting by themselves, but the present book is big enough!

A most active field of recent research is concerned with quantum integrable systems or the closely related field of exactly soluble models in statistical mechanics. When writing this book we always had the quantum theory present in mind, and have introduced all classical objects which have a well-known quantum counterpart, or are semi-classical limits of quantum objects. This explains our emphasis on Hamiltonians methods, Poisson brackets, classical r-matrices, Lie–Poisson properties of dressing transformations and the method of separation of variables. Although there is nothing quantum in this book, a large part of the apparatus necessary to understand the literature on quantum integrable systems is in fact present.

The bibliography for integrable systems would fill a book by itself. We have made no attempt to provide one. Instead, we give, at the end of each chapter, a short list of references, which complements and enhances the material presented in the chapter, and we highly encourage the reader to consult them. Of course these references are far from complete, and we apologize to the numerous authors having contributed to the domain, and whose due credit is not acknowledged. Finally we want to thank our many colleagues from whom we learned so much and with whom we have discussed many parts of this book.

2
Integrable dynamical systems

We introduce the definition of integrable systems through the Liouville theorem, i.e. systems for which n conserved quantities in involution are known on a phase space of dimension $2n$. The Liouville theorem asserts that the equations of motion can then be solved by quadrature. The notion of Lax matrix is introduced. This is a matrix whose elements are dynamical and whose time evolution is isospectral, a central object in the theory. It is also shown that the Poisson brackets of the elements of the Lax matrix are expressed in the so-called r-matrix form. Finally, we present some historical examples of integrable systems which are solved by the method of separation of variables. This leads to linearization of the time evolution on the Jacobian of Riemann surfaces, another recurring theme in the book.

2.1 Introduction

In Classical Mechanics the state of the system is specified by a point in phase space. This is generally a space of even dimension with coordinates of position q_i and momentum p_i. The Hamiltonian is a function on phase space, denoted $H(p_i, q_i)$. The equations of motion are a first order differential system taking the Hamiltonian form:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.1}$$

Here and in the following, a dot will refer to a time derivative. For any function $F(p, q)$ on phase space, this implies that $F(p(t), q(t))$ obeys:

$$\dot{F} \equiv \frac{dF}{dt} = \{H, F\}$$

where for any functions F and G the Poisson bracket $\{F, G\}$ is defined as:

$$\{F, G\} \equiv \sum_i \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}$$

For the coordinates p_i, q_i themselves we have

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij} \tag{2.2}$$

The quantity $H(p, q)$ is automatically conserved under time evolution, $\frac{d}{dt}H(p, q) = 0$, so that the motion takes place on the subvariety of phase space defined by $H = E$ constant.

Historically, it proved very difficult to find dynamical systems such that eqs. (2.1) could be solved exactly. However, there is a general framework where the explicit solutions can be obtained by solving a finite number of algebraic equations and computing finite number of integrals, i.e. the solution is obtained by quadratures. These dynamical systems are the Liouville integrable systems that we will consider in this book. A dynamical system on a phase space of dimension $2n$ is Liouville integrable if one knows n independent functions F_i on the phase space which Poisson *commute*, that is $\{F_i, F_j\} = 0$. The Hamiltonian is assumed to be a function of the F_i.

In order to understand the geometry of the situation, let us discuss a very simple example: the harmonic oscillator. The phase space is of dimension 2 and the Hamiltonian is $H = \frac{1}{2}(p^2+\omega^2 q^2)$ with Poisson bracket $\{p, q\} = 1$. The phase space is fibred into ellipses $H = E$ except for the point $(0, 0)$ which is a stationary point. An adapted coordinate system ρ, θ is given by:

$$p = \rho\cos(\theta), \quad q = \frac{\rho}{\omega}\sin(\theta) \tag{2.3}$$

and the non-vanishing Poisson bracket is $\{\rho, \theta\} = \omega/\rho$. In these coordinates the flow reads:

$$\rho = \sqrt{2E}, \quad \theta = \omega t + \theta_0$$

i.e. the flow takes place on the above ellipsis.

This can be straightforwardly generalized to a direct sum of n harmonic oscillators with

$$H = \sum_{i=1}^{n} \frac{1}{2}(p_i^2 + \omega_i^2 q_i^2)$$

and Poisson bracket eq. (2.2). We do have n conserved quantities in involution, $F_i = \frac{1}{2}(p_i^2 + \omega_i^2 q_i^2)$, and the level manifold M_f, i.e. the set of points of phase space such that $F_i = f_i$, is an n-dimensional real torus, which is explicitly a cartesian product of n topological circles. The motion takes place on these tori which foliate the phase space. We can intoduce n angles θ_i as above which evolve linearly in time with frequency ω_i. An orbit of the dynamical flow is dense on the torus when the ω_i are rationally independent.

For Liouville integrable systems, we shall assume that the conserved quantities are well-behaved so that the n dimensional surfaces M_f defined by $F_i = f_i$ are generically regular, and foliate the phase space. This does not preclude the existence of singular points such as $p_i = q_i = 0$ in the above example of the harmonic oscillator. In this setting we are now going to prove the Liouville theorem and show that the geometry of the situation is analogous to that of the harmonic oscillator example.

2.2 The Liouville theorem

We consider a dynamical Hamiltonian system with phase space M of dimension $2n$. Introduce canonical coordinates p_i, q_i such that the non-degenerate Poisson bracket reads as in eq. (2.2). As usual a non-degenerate Poisson bracket on M is equivalent to the data of a non-degenerate closed 2-form ω, $d\omega = 0$, defined on M, called the symplectic form, see Chapter 14. In the canonical coordinates the symplectic form reads

$$\omega = \sum_j dp_j \wedge dq_j$$

Let H be the Hamiltonian of the system.

Definition. *The system is Liouville integrable if it possesses n independent conserved quantities F_i, $i = 1, \ldots, n$, $\{H, F_j\} = 0$, in involution*

$$\{F_i, F_j\} = 0$$

The independence means that at generic points (i.e. anywhere except on a set of measure zero), the dF_i are linearly independent, or that the tangent space of the surface $F_i = f_i$ exists everywhere and is of dimension n. There cannot be more than n independent quantities in involution otherwise the Poisson bracket would be degenerate. In particular, the Hamiltonian H is a function of the F_i.

The Liouville theorem. *The solution of the equations of motion of a Liouville integrable system is obtained by "quadrature".*

Proof. Let $\alpha = \sum_i p_i dq_i$ be the canonical 1-form and $\omega = d\alpha = \sum_i dp_i \wedge dq_i$ be the symplectic 2-form on the phase space M. We will construct a canonical transformation $(p_i, q_i) \to (F_i, \Psi_i)$ such that the conserved quantities F_i are among the new coordinates:

$$\omega = \sum_i dp_i \wedge dq_i = \sum_i dF_i \wedge d\Psi_i$$

If we succeed in doing that, the equations of motion become trivial:

$$\begin{aligned} \dot{F}_j &= \{H, F_j\} = 0 \\ \dot{\psi}_j &= \{H, \psi_j\} = \frac{\partial H}{\partial F_j} = \Omega_j \end{aligned} \tag{2.4}$$

The Ω_j depend only on F and so are constant in time. In these coordinates, the solution of the equations of motion read:

$$F_j(t) = F_j(0), \quad \psi_j(t) = \psi_j(0) + t\Omega_j$$

To construct this canonical transformation, we exhibit its so-called generating function S. Let M_f be the level manifold $F_i(p, q) = f_i$. Suppose that on M_f we can solve for p_i, $p_i = p_i(f, q)$, and consider the function

$$S(F, q) \equiv \int_{m_0}^{m} \alpha = \int_{q_0}^{q} \sum_i p_i(f, q) dq_i$$

where the integration path is drawn on M_f and goes from the point of coordinate $(p(f, q_0), q_0)$ to the point $(p(f, q), q)$, where q_0 is some reference value.

Suppose that this function exists, i.e. if it does not depend on the path from m_0 to m, then $p_j = \frac{\partial S}{\partial q_j}$. Defining ψ_j by

$$\psi_j = \frac{\partial S}{\partial F_j}$$

we have

$$dS = \sum_j \psi_j dF_j + p_j dq_j$$

Since $d^2 S = 0$ we deduce that $\omega = \sum_j dp_j \wedge dq_j = \sum_j dF_j \wedge d\psi_j$. This shows that if S is a well-defined function, then the transformation is canonical.

To show that S exists, we must prove that it is independent of the integration path. By Stokes theorem, we have to prove that:

$$d\alpha|_{M_f} = \omega|_{M_f} = 0$$

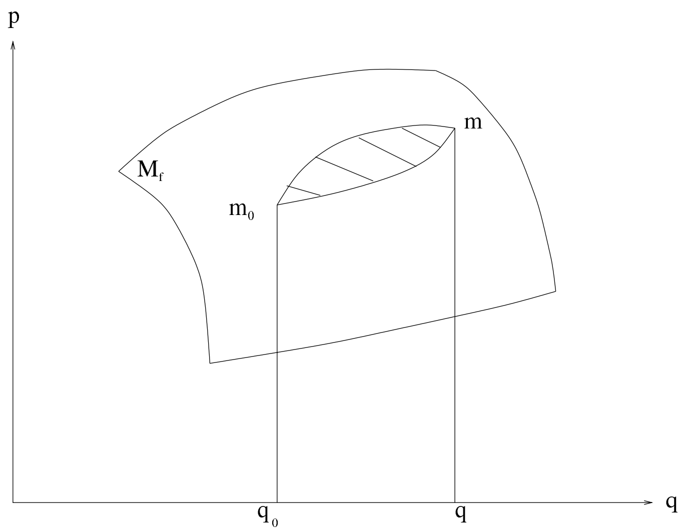

Fig. 2.1. A leaf M_f on phase space

Let X_i be the Hamiltonian vector field associated with F_i, defined by $dF_i = \omega(X_i, \cdot)$,

$$X_i = \sum_k \frac{\partial F_i}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial F_i}{\partial p_k} \frac{\partial}{\partial q_k}$$

These vector fields are tangent to the manifold M_f because the F_j are in involution,

$$X_i(F_j) = \{F_i, F_j\} = 0$$

Since the F_j are assumed to be independent functions, the tangent space to the submanifold M_f is generated at each point $m \in M$ by the vectors $X_i|_m \quad (i = 1, \ldots, n)$. But then $\omega(X_i, X_j) = dF_i(X_j) = 0$ and we have proved that $\omega|_{M_f} = 0$, and therefore S exists. ■

We have effectively obtained the solution of the equations of motion through one quadrature (to calculate the function S) and some "algebraic manipulation" (to express the p as functions of q and F)

Remark 1. From the closedness of α on M_f, the function S is unchanged by *continuous* deformations of the path (m_0, m). However, if M_f has non-trivial cycles, which is generically the case, S is a multivalued function defined in a neighbourhood of M_f. The variation over a cycle

$$\Delta_{\text{cycle}} S = \int_{\text{cycle}} \alpha$$

is a function of F only. This induces a multivaluedness of the variables ψ_j: $\Delta_{\text{cycle}}\psi_j = \frac{\partial}{\partial F_j}\Delta_{\text{cycle}} S$. For instance, in the case of harmonic oscillators, we see that above each point $(q_1, \ldots, q_n)$ we have 2^n points on the M_f level surface, due to the independent choices of sign in $p_i = \pm\sqrt{2f_i - \omega_i^2 q_i^2}$. So we have many choices for the path of integration, reflecting the topology of the torus.

Remark 2. The definition we have given of a Liouville integrable system requires some care. Given any Hamiltonian H, the Darboux theorem, see Chapter 14, implies that we can always find *locally* a system of canonical coordinates on phase space $(P_1, \ldots, P_n; Q_1, \ldots, Q_n)$, with $H = P_1$, hence fulfilling the hypothesis of the Liouville theorem. For integrable systems we require that the conserved quantities are globally defined on a sufficiently large open set, and that the surfaces $F_i = f_i$ are well-behaved and foliate the phase space. This is not generally the case for the P_i constructed by the Darboux theorem. Moreover, in all known examples, the conserved quantities are even algebraic functions of canonical coordinates on some open domain and the solutions of the equations of motion are analytic.

Remark 3. Using the Poisson commuting functions F_i, one can solve *simultaneously* the n time evolution equations $dF/dt_i = \{F_i, F\}$, since:

$$\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}F - \frac{\partial}{\partial t_j}\frac{\partial}{\partial t_i}F = \{F_i, \{F_j, F\}\} - \{F_j, \{F_i, F\}\} = \{\{F_i, F_j\}, F\} = 0$$

Since the Hamiltonian vector fields are well-defined and linearly independent everywhere, the flows define a locally free (no fixed points) and transitive (goes everywhere) action of a small open set in $\mathbb{R}^n$ on the surface M_f. Assuming that M_f is connected and compact, the flows extend to all values of the times t_i and fill the whole surface M_f, hence we have a surjective action of $\mathbb{R}^n$ on M_f. The stabilizer of a point is an Abelian discrete subgroup of $\mathbb{R}^n$ since the action is locally free, so it is of the form $\mathbb{Z}^n$. Thus M_f appears as the quotient of $\mathbb{R}^n$ by $\mathbb{Z}^n$, i.e. a torus. This refinement, due to Arnold, of the Liouville theorem shows that, under suitable global hypothesis, the phase space is indeed foliated by n dimensional tori, called the Liouville tori. It is remarkable that for small perturbations of integrable systems, there still exist Liouville tori "almost everywhere". This is the content of the famous Kolmogorov–Arnold–Moser (KAM) theorem.

2.3 Action–angle variables

As already noticed in the proof of the Liouville theorem, the level manifold M_f has non-trivial cycles. Under suitable compactness and connectivity conditions, the M_f are n-dimensional tori T_n. This points to the introduction of angle variables to describe the motion along the cycles. The torus T_n is isomorphic to a product of n circles C_i. We may choose special angular coordinates on M_f dual to the n fundamental cycles C_i (see eq. (2.5)).