INTRODUCTION TO GEOMAGNETIC FIELDS

Second Edition

WALLACE H. CAMPBELL

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 1997
Second edition 2003

Printed in the United Kingdom at the University Press, Cambridge

Typefaces Times Roman 10/13 pt and Stone Sans *System* $\LaTeX 2_\varepsilon$ [TB]

A catalog record for this book is available from the British Library

Library of Congress Cataloging in Publication data

ISBN 0 521 82206 8 hardback
ISBN 0 521 52953 0 paperback

Contents

Chapter 1
The Earth's main field

1.1 Introduction

The science of geomagnetism developed slowly. The earliest writings about compass navigation are credited to the Chinese and dated to 250 years B.C. (Figure 1.1). When Gilbert published the first textbook on geomagnetism in 1600, he concluded that the Earth itself behaved as a great magnet (Gilbert, 1958 reprint) (Figure 1.2). In the early nineteenth century, Gauss (1848) introduced improved magnetic field observation techniques and the spherical harmonic method for geomagnetic field analysis. Not until 1940 did the comprehensive textbook of Chapman and Bartels bring us into the modern age of geomagnetism. The bibliography in the Appendix, Section B.7, lists some of the major textbooks about the Earth's geomagnetic field that are currently in use.

For many of us the first exposure to the concept of an electromagnetic field came with our early exploration of the properties of a magnet. Its strong attraction to other magnets and to objects made of iron indicated immediately that something special was happening in the space between the two solid objects. We accepted words such as *field, force field*, and *lines of force* as ways to describe the strength and direction of the push or pull that one magnetic object exerted on another magnetic material that came under its influence. So, to start our subject, I would like to recall a few of our experiences that give reality to the words *magnetic field* and *dipole field*.

Toying with a couple of bar magnets, we find that they will attract or oppose each other depending upon which ends are closer. This experimentation leads us to the realization that the two ends of a magnet have

Figure 1.1. The Chinese report that the compass (Si Nan) is described in the works of Hanfucious, which they date between 280 and 233 B.C. The spoon-shaped magnetite indicator, balancing on its heavy rounded bottom, permits the narrow handle to point southward, to align with the directions carved symmetrically on a nonmagnetic baseplate. This photograph shows a recent reproduction, manufactured and documented by the Central Iron and Steel Research Institute, Beijing.

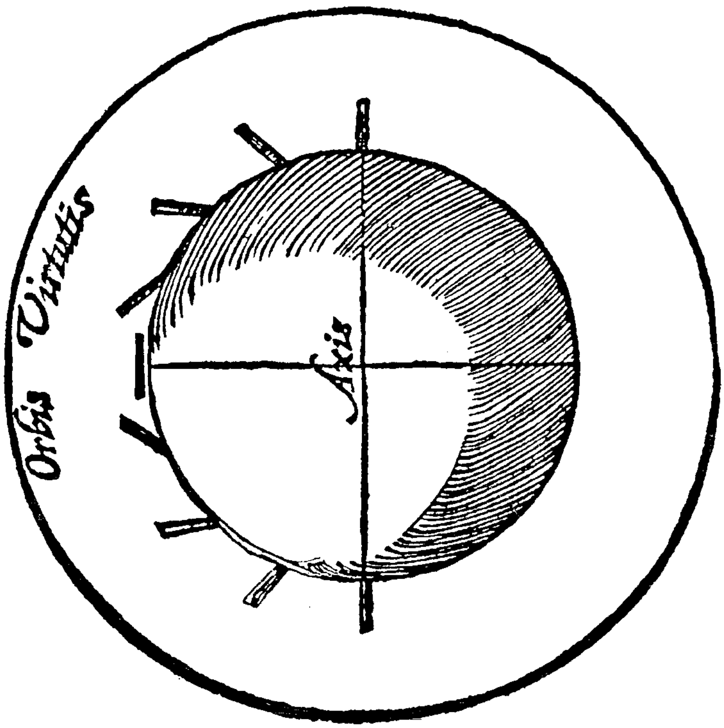


Figure 1.2. Diagram from Gilbert's 1600 textbook on geomagnetism in which he shows that the Earth behaves as a great magnet. The field directions of a dip-needle compass are indicated as tilted bars.

oppositely directed effects or *polarity*. It is a short but easy step for us to understand the operation of a compass when we are told that the Earth behaves as a great magnet.

In school classrooms, at more sophisticated levels of science exploration, many of us learned that positive and negative electric charges also have attraction/repulsion properties. The simple arrangement of two charges of opposite sign constitute an *electric dipole*. The product of the charge size and separation distance is called the *dipole moment*; the pattern of the resulting electric field is called *dipolelike*.

The subject of this chapter is the dipolelike *magnetic field* that we call the "Earth's main field." I will demonstrate that this magnetic field shape is similar in form to the field from a pair of electric charges with opposite signs. Knowing that a loop of current could produce a dipolelike field, arguments are given to discount the existence of a large, solid iron magnet as the Earth-field source. Rather, the main field origin seems to reside with the currents flowing in the outer liquid core of the Earth, which derive their principal alignment from the Earth's axial spin.

To bring together the many measurements of magnetic fields that are made on the Earth, a method has been developed for depicting the systematic behavior of the fields on a spherical surface. This spherical harmonic analysis (SHA) creates a mathematical representation of the entire main field anywhere on Earth using only a small table of numbers. The SHA is also used to prove that the main field of the Earth originates mostly from processes interior to the surface and that only a minor proportion of the field arises from currents in the high and distant external environment of the Earth. We shall see that the SHA divides the contributions of field into dipole, quadrupole, octupole, etc., distinct parts. The largest of these, the dipole component, allows us to fix a geomagnetic coordinate system (overlaying the geographic coordinates) that helps researchers easily organize and explain various geophysical phenomena.

The slow changes of flow processes in the Earth's deep liquid interior that drive the geomagnetic field require new sets of SHA tables and revised geomagnetic maps to be produced regularly over the years. Such changes are typically quite gradual so that some of the past and future conditions are predictable over a short span of time. A special science of paleomagnetism examines the behavior of the ancient field before the Earth assumed its present form. Paleomagnetic field changes, for the most part, are not predictable and give evidence of the magnetism source region and the Earth's evolution.

Also, in this chapter we will look at the definitions of terms used to represent the Earth's main field. We will see how the descriptive maps and field model tables are obtained and appreciate the meaning that these numbers provide for us.

1.2 Magnetic Components

A typical inexpensive compass such as a small needle dipole magnet freely balanced, or suspended at its middle by a long thread, will align itself with the local horizontal magnetic field in a general north–south direction. The north-pointing end of this magnet is called the *north pole*; its opposite end, the *south pole*. Because opposite ends of magnets, or compass needles, are found to attract each other, the Earth's dipole field, attracting the north pole of a magnet toward the northern arctic region, should really be called a south pole. Fortunately, to avoid such confusion, the convention is ignored for the Earth so that geographic and geomagnetic pole names agree. Other adjectives sometimes given are *Boreal* for the northern pole and *Austral* for the southern pole. We say that our compass points northward, although, in fact, it just aligns itself in the north–south direction. The early Chinese, who first used a compass for navigation (at least by the fourteenth century) considered southward to be the important pointing direction (Figure 1.1). Naturally magnetized magnetite formed the first compasses. Early Western civilization called that black, heavy iron compound *lodestone* (sometimes spelled *loadstone*) meaning "leading stone." It is believed that the word "magnet" is derived from Magnesia (north-east of Ephesus in ancient Macedonia) where lodestone was abundant.

By international agreement, a set of names and symbols is used to describe the Earth's field components in a "right-hand system." Figure 1.3 illustrates this nomenclature for a location in the Northern Hemisphere where the total field vector points into the Earth. The term *right-hand system* means that if we aligned the thumb and first two fingers of our right hand with the three edges that converge at a box corner, then the x direction would be indicated by our thumb, the y direction by our index (pointing) finger, and the z direction by the remaining finger. We say these are the three *orthogonal directions* along the X, Y, and Z axes in space because they are at right angles (90°) to each other. When a measurement has both a size (magnitude) and a direction, it can be drawn as an arrow with a particular heading that extends a fixed distance (to indicate magnitude) from the origin of an orthogonal coordinate system. Such an arrow is called a *vector* (see Section A.6). Any vector may be represented in space by the composite vectors of its three orthogonal components (projections of the arrow along each axis).

A magnetic field is considered to be in a positive direction if an isolated north magnetic pole would freely move in that field direction.

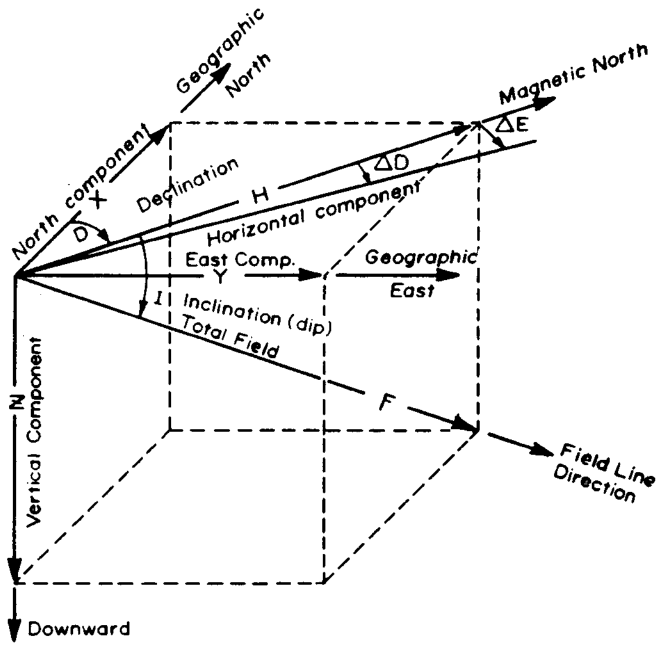


Figure 1.3. Components of the geomagnetic field measurements for a sample Northern Hemisphere total field vector **F** inclined into the Earth. An explanation of the letters and symbols is given in the text.

Observers prefer to describe a vector representing the Earth's field in one of two ways: (1) three orthogonal component field directions with positive values for geographic northward, eastward, and vertical into the Earth (negative values for the opposite directions) or (2) the horizontal magnitude, the eastward (minus sign "–" for westward) angular direction of the horizontal component from geographic northward, and the downward (vertical) component. The first set is typically called the *X, Y*, and *Z* (*XYZ-component*) representation; the last set is called the *H* (horizontal), *D* (declination), and *Z* (into the Earth) (*HDZ-component*) representation (or sometimes *DHZ*). In equations, a boldface on a field letter (e.g., **H**) will be used to emphasize the vector property; without the boldface we will just be interested in the size (magnitude).

In the early days of sailing-ship navigation the important measurement for ship direction was simply *D*, the angle between true north and the direction to which the compass needle points. Ancient magnetic observations therefore used the *HDZ* system of vector representation. By simple geometry we obtain

$$X = H\cos(D), \quad Y = H\sin(D). \tag{1.1}$$

(See Section A.5 for trigonometric functions.) The total field strength, *F* (or *T*), is given as

$$F = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{H^2 + Z^2}. \tag{1.2}$$

The angle that the total field makes with the horizontal plane is called the *inclination, I*, or *dip angle*:

$$\frac{Z}{H} = \tan(I). \tag{1.3}$$

The quiet-time annual mean inclination of a station, called its "Dip Latitude", becomes particularly important for the ionosphere at about 60 to 1000 km altitude (Section 2.3) where the local conductivity is dependent upon the field direction.

Although the *XYZ* system provides the presently preferred coordinates for reporting the field and the annual INTERMAGNET data disks follow this system, activity-index requirements (Section 3.13) and some national observatories publish the field in the *HDZ* system. It is a simple matter to change these values using the angular relationships shown in Figure 1.3. The conversion from X and Y to H and D becomes

$$H = \sqrt{(X^2 + Y^2)} \text{ and } D = \tan^{-1}(Y/X). \tag{1.4}$$

On occasion, the declination angle D in degrees (D°) is expressed in magnetic eastward directed field strength D (nT) and obtained from the relationship

$$D\,(\mathrm{nT}) = H \tan(D^\circ). \tag{1.5}$$

Sometimes the change of D (nT) about its mean is called a magnetic eastward field strength, ΔE. (For small, incremental changes in a value it is the custom to use the symbol Δ.)

In the Earth's spherical coordinates, the three important directions are the angle (θ) measured from the geographic North Pole along a great circle of longitude, the angle (ϕ) eastward along a latitude line measured from a reference longitude, and the radial direction, r, measured from the center of the Earth. On the Earth's surface (where x, y, and z correspond to the $-\theta$, ϕ, and $-r$ directions) the field, **B**, in spherical coordinates becomes

$$B_\theta = -X, \quad B_\phi = Y, \quad \text{and} \quad B_r = -Z. \tag{1.6}$$

Originally, the *HDZ* system was used at most world observatories because the measuring instruments were suspended magnets and there was a direct application to navigation and land survey. Usually, only an angular reading between a compass northward direction and geographic north was needed. In the *HDZ* system, the data from different observatories have different component orientations with respect to the Earth's axis and equatorial plane. The $\theta\phi r$ system is used for mathematical treatments in spherical analysis (of which we will see more in this chapter). The XYZ coordinate system is necessary for field recordings by many high-latitude observatories because of the great disparity in the geographic angle toward magnetic north at polar region sites. The XYZ system is becoming the preferred coordinate system for most modern digital observatories. Computers have made it simple to interchange the digital field representation into the three coordinate systems. Figure 1.3 shows the angle of *inclination* (*dip*), I, and the *total field* vector, **F**.

The unit size of fields is a measurable quantity. We can appreciate this fact when we consider the amount of force needed to separate magnets of different strengths or the amount of force that must be used to push a compass needle away from its desired north–south direction. Let us not elaborate on tedious details of establishing the unit sizes of fields. What will be called "field strength" results from a measurement of a quantity called "magnetic flux density," B, that can be obtained from a comparison to force measurements under precisely prescribed conditions. The units for this field strength have appeared differently over the years; Table 1.1 lists equivalent values of B.

Table 1.1. *Equivalent magnetic field units*

$B = 10^4$ Gauss
$B = 1$ Weber/meter^2
$B = 10^9$ gamma
$B = 1$ Tesla

At present, in most common usage, the convenient size of magnetic field units is the gamma, or γ, a lower-case Greek letter to honor Carl Friedrich Gauss, the nineteenth-century scientist from Göttingen, Germany, who contributed greatly to our knowledge of geomagnetism. The International System (SI) of units, specified by an agreement of world scientists, recommends use of the Tesla (the name of an early pioneer in radiowave research). With the prefix nano meaning 10^{-9}, of course, one gamma is equivalent to one nanotesla (nT), so there should be no confusion when we see either of these expressions. To familiarize the reader with this interchange (which is common in the present literature), I will use either name at different times in this book.

Geomagnetic phenomena have a broad range of scales. The main field is nearly $60,000$ (6×10^4; see scientific notation in Section A.3) gamma near the poles and about $30,000$ (3×10^4) gamma near the equator. A small, 2 cm, calibration magnet I have in my office is 1×10^8 gamma at its pole (about 10,000 times the Earth's surface field in strength). Quiet-time daily field variations can be about 20 gamma at midlatitudes and 100 gamma at equatorial regions. Solar–terrestrial disturbance–time variations occasionally reach 1,000 gamma at the auroral regions and 250 gamma at midlatitudes. Geomagnetic pulsations arising in the Earth's space environment are measured in the 0.01 gamma to 10 gamma range at surface midlatitude locations. In Chapter 4 we will see how this great 10^6 dynamic range of the source fields is accommodated by the measuring instruments.

The magnetic fields that interest us arise from currents. Currents come from charges that are moving. Much of the research in geomagnetism concerns the discovery (or the use of) the source currents responsible for the fields found in the Earth's environment. Then, we ask, what about the fields from magnetic materials; where is the current to be found? A simple "Bohr model" (with planetarylike electrons about a sunlike nucleus) suffices in our requirements for visualizing the atomic structure. In this model the spinning charges of orbital electrons in the atomic structure provide the major magnetic properties. Most atoms in nature contain even numbers of orbiting electrons, half circulating in one direction, half in another, with both their orbital and spin magnetic effects canceling. When canceling does not occur, typically when there are unpaired electrons, there is a tendency for the spins of adjacent atoms or molecules to align, establishing a domain of unique field direction. Large groupings of similar domains give a magnet its special properties. We will discuss this subject further in Section 4.2 on geomagnetic instruments. However, for now, we find consistency in the idea that charges-in-motion create our observed magnetic fields.

1.3 Simple Dipole Field

To many of us, the first exposure to the term "dipole" occurred in learning about the electric field of two point charges of opposite sign placed a short distance from each other. Figure 1.4 represents such an arrangement of charges, $+q$ and $-q$ (whose sizes are measured in units called "coulombs"), separated by a distance, d, along the z axis of an orthogonal coordinate system. We call the value (qd) by the distinctive name *dipole moment* and assign it the letter "p." The units of p are coulomb-meters. Figure 1.5 shows the dipole as well as symmetric quadrupole and octupole arrangements of charge at the corners of the respective figures. The reason for introducing the electric charges and multipoles here is to help us understand the nomenclature of the magnetic fields, for which isolated poles do not exist, although the magnetic field shapes are identical to the shapes of multipole electric fields.

The point $P(x, y, z)$ is the location, for position x, y, and z from the dipole axis origin, at which the electric field strength from the dipole charges is to be determined (Figure 1.4). This location is a distance

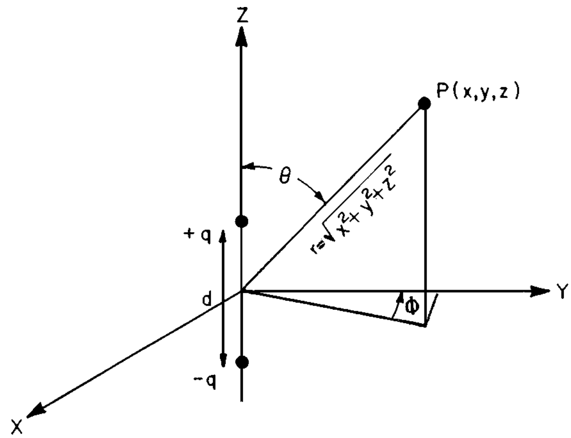


Figure 1.4. Electric dipole of charges, $\pm q$, and the corresponding coordinate system. An explanation of the letters and symbols is given in the text.

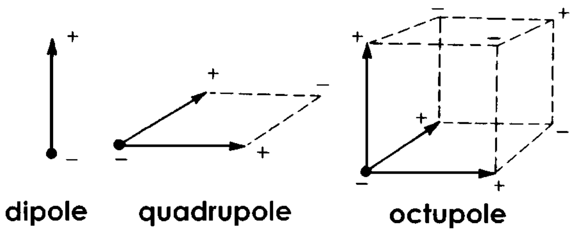


Figure 1.5. Charge distribution for the electric dipole, quadrupole, and octupole configurations.

$r = \sqrt{x^2 + y^2 + z^2}$ from the midpoint between the two charges and at an angle θ from the positive Z axis. We will call this angle the colatitude (colatitude $= 90° -$ latitude) of a location. The angle to the projection of r onto the X–Y plane, measured clockwise, is called ϕ. Later we will identify this angle with east longitude. Sometimes we will see the letter e, with the r, θ, or ϕ subscript, used to indicate the respective unit directions in spherical coordinates. Obviously, there is symmetry about the Z axis so the electric field at P doesn't change with changes in ϕ. Thus it will be sufficient to describe the components of the dipole field simply along r and θ directions.

Now I will need to use some mathematics. It is necessary to show the exact description that defines something we can easily visualize: the shape of an electric field resulting from two electric charges of opposite sign and separated by a small distance. I will then demonstrate that such a mathematical representation is identical to the form of a field from a current flowing in a circular loop. That proof is important for all our descriptions of the Earth's main field and its properties because we will need to discuss the source of the main dipole field, global coordinate systems, and main-field models. If the mathematics at this point is too difficult, just read it lightly to obtain the direction of the development and come back to the details when you are more prepared.

Let us start with a property called the *electric potential*, Ω, of a point charge in air from which we will subsequently obtain the electric field.

$$\Omega = \frac{q}{4\pi \in_0 r}, \tag{1.7}$$

where q is the coulomb charge, r is the distance in meters to the observation, $\in_0$ (the "inductive capacity of free space") is a constant typical of the medium in which the field is measured, and Ω is measured in volts.

For two charges with opposite signs, separated by a distance, d, the potential at point r at x, y, z coordinate distances becomes

$$\Omega = \frac{1}{4\pi \in_0}\left[\frac{q}{\sqrt{(z-d/2)^2+x^2+y^2}} + \frac{-q}{\sqrt{(z+d/2)^2+x^2+y^2}}\right] \tag{1.8}$$

Now, for the typical dipole, d is very small with respect to r so, with some algebraic manipulation, we can write

$$\Omega = \frac{q}{4\pi \in_0 r}\left[\left(1+\frac{zd}{2r^2}\right) - \left(1-\frac{zd}{2r^2}\right)\right] + \Delta A, \tag{1.9}$$

where ΔA represents terms that become negligible when $d = r$. Because $(z/r) = \cos(\theta)$, Equation (1.9) can be written in the form

$$\Omega = \frac{qd\cos(\theta)}{4\pi \in_0 r^2}. \tag{1.10}$$

Now let us see the form of the electric field using Equation (1.7). We are going to be interested in a quantity called the *gradient* or *grad* (represented by an upside-down Greek capital delta; see Section A.8) of the potential. In spherical coordinates the gradient can be represented by the derivatives (slopes) in the separate coordinate directions:

$$\nabla\Omega = \mathbf{e}_r\left(\frac{\delta\Omega}{\delta r}\right) + \mathbf{e}_\theta\left(\frac{\delta\Omega}{r\delta\theta}\right) + \mathbf{e}_\phi\left(\frac{1}{r(\sin\theta)}\frac{\delta\Omega}{\delta\phi}\right), \tag{1.11}$$

where the **e**s are the unit vectors in the three spherical coordinate directions, r, θ, and ϕ. The electric field, obtained from the negative of that gradient, is given as

$$\mathbf{E}_r = -\frac{\delta\Omega}{\delta r} = \frac{p}{2\pi \in_0}\left(\frac{\cos\theta}{r^3}\right)\mathbf{e}_r \tag{1.12}$$

and

$$\mathbf{E}_\theta = -\frac{1}{r}\frac{\delta\Omega}{\delta\theta} = \frac{p}{4\pi \in_0}\left(\frac{\sin\theta}{r^3}\right)\mathbf{e}_\theta, \tag{1.13}$$

where p is the electric dipole moment qd.

Symmetry about the dipole axis means Ω doesn't change with angle ϕ, so **E** in the ϕ direction is zero. Equations (1.12) and (1.13) define the form of an electric dipole field strength in space. If we would like to draw lines representing the shape of this dipole field (to show the directions

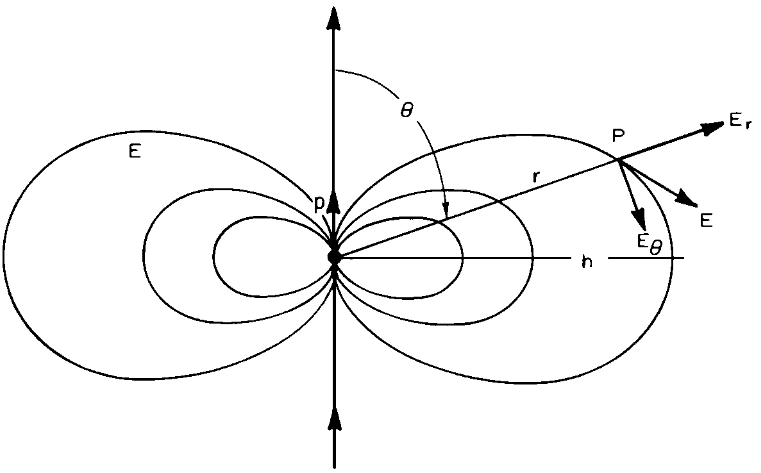


Figure 1.6. Electric dipole (of moment *p*) field configuration with directions for the components of electric field vectors **E** in the *r* and *θ* directions to an arbitrary observation point *P*; *h* is the equatorial field line distance.

that a charge would move in its environment), there is a convenient equation,

$$r = h\sin^2(\theta), \tag{1.14}$$

that can be used, in which h is the distance from the dipole center to the equatorial crossing (at $\theta = 90°$) of the field line (Figure 1.6).

These field descriptions (Equations (1.11) to (1.13)) come from scientists, mainly of the late eighteenth and early nineteenth centuries, who interpreted laboratory measurements of charges, currents, and fields to establish mathematical descriptions of the natural electromagnetic "laws" they observed. At first there was a multitude of laws and equations, covering many situations of currents and charges and relating electricity to magnetism. Then by 1873, James Clerk Maxwell brought order to the subject by demonstrating that all the "laws" could be derived from a few simple equations (that is, "simple" in mathematical form). For example, in a region where there is no electric charge, Maxwell's equations show that there is no "divergence" of electric field, a statement that mathematical shorthand shows as

$$\nabla \cdot \mathbf{E} = 0, \tag{1.15}$$

where the "del-dot" symbol is explained in Section A.8. But **E** is given as – grad Ω (which is titled "the negative gradient of the scalar potential"). Thus, for the mathematically inclined, it follows that

$$\nabla \cdot \mathbf{E} = -\nabla\nabla\Omega = 0 \tag{1.16}$$

or

$$\nabla^2\Omega = 0, \tag{1.17}$$

for which Equation (1.10) can be shown (by those skilled in mathematical manipulations) to be a solution for a dipole configuration of charges.

A simple experiment, often duplicated in science classrooms, is to connect a battery and an electrical resistor to the ends of an iron wire (with an insulated coating) that has been wrapped in a number of turns, spiraling about a wooden matchstick for shape. It is then demonstrated that when current flows, the wire helix behaves as if it were a dipole magnet aligned with the matchstick, picking up paper clips or deflecting a compass needle. If the current direction is reversed by interchanging the battery connections, then the magnetic field direction reverses.

Now let us illustrate with mathematics how a current flowing in a simple wire loop produces a magnetic field in the same form as the electric dipole. My purpose is to help us visualize a magnetic dipole, when there isn't a magnetic substance corresponding to the electric charges, so that we can later understand the origin of the main field in the liquid flows of the Earth's deep-core region.

Consider Figure 1.7, in which a current, **i**, is flowing in the X–Y plane along a loop enclosing area, A, of radius b, for which Z is the normal (perpendicular) direction. Let P be any point at a distance, r, from the loop center and at a distance, R, from a current element moving a distance, ds. The electromagnetic law for computing the element of field, dB, from the current along the wire element, ds, is

$$dB = \frac{\mu_0 i}{4\pi} \frac{ds\, R \sin(\alpha)}{R^3}, \tag{1.18}$$

where α is the angle between ds and R, so that dB is in the direction that

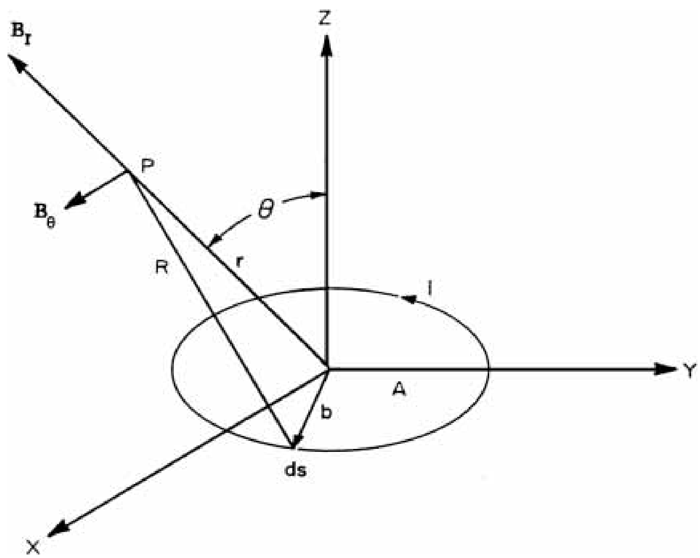


Figure 1.7. Coordinate system for a loop of current i, having radius b, area A, and enclosed perimeter of element length ds. The magnetic field vectors of **B** in the r and θ directions with respect to an orthogonal (right-hand) x, y, z coordinate system are shown.

a right-handed screw would move when turning from *ds* (in the current direction) toward *R* (directed toward *P*). The magnetic properties of the medium are indicated by the constant, μ_0, called the "permeability of free space." We wish to find the *r* and θ magnetic field components of **B**, at any point in space about the loop, with the simplifying conditions that $r \gg b$. Using the electromagnetic laws, we sum the *dB* contributions to the field for each element of distance around the loop, and after some mathematics obtain

$$B_r = \frac{\mu_0 i A \cos(\theta)}{2\pi r^3} \tag{1.19}$$

and

$$B_\theta = \frac{\mu_0 i A \sin(\theta)}{4\pi r^3}. \tag{1.20}$$

Comparing these two field representations with those we obtained for the electric dipole (Equations (1.12) and (1.13)), we see that the same field forms will be produced if we let the current times the area (iA) correspond to *p*, the electric dipole moment, *qd*. Thus, calling *M* the *magnetic dipole moment*,

$$M = iA \tag{1.21}$$

or

$$M = md, \tag{1.22}$$

where *d* becomes the equivalent separation of hypothetical magnetic poles of strength *m*.

We saw, in the parallel case of the electric dipole, that *E* was obtained from the scalar potential in a charge-free region. In a similar fashion, Maxwell's equations show that

$$\mathbf{B} = -\nabla V, \tag{1.23}$$

where *V* is called the magnetic scalar potential. Then, a person with math competence can write, for a current-free region (where curl $\mathbf{B} = 0$),

$$\nabla^2 V = 0 \tag{1.24}$$

and obtain a dipole solution

$$V = \frac{\mu_0 M \cos(\theta)}{4\pi r^2}. \tag{1.25}$$

To a first approximation, the Earth's field in space behaves as a magnetic dipole. At the Earth's surface we call $r = a$. Then

$$B_r = -Z = \frac{2[\mu_0 M \cos(\theta)]}{4\pi a^3} = Z_0 \cos(\theta) \tag{1.26}$$

and

$$B_\theta = -H = \frac{\mu_0 M \sin(\theta)}{4\pi a^3} = H_0 \sin(\theta), \tag{1.27}$$

where constant $H_0 = Z_0/2$. The total field magnitude, F, is just

$$F = \sqrt{H^2 + Z^2}. \tag{1.28}$$

On average, about ninety percent of the Earth's field is dipolar so we can use the approximation, $H_0 = 3.1 \times 10^4$ gamma, for rough field modeling. In Equations (1.26) and (1.27), recall (Section A.5) that $\sin(90^\circ) = 1$, $\sin(0^\circ) = 0$, $\cos(90^\circ) = 0$, and $\cos(0^\circ) = 1$. For the Southern Hemisphere, where $90^\circ < \theta \leq 180^\circ$, note that $\sin(180^\circ - \theta) = \sin(\theta)$ and $\cos(180^\circ - \theta) = -\cos(\theta)$. So the magnitude of the Earth's field at the equator ($\theta = 90^\circ$) is just H_0 and at the poles ($\theta = 0^\circ$ or 180°) just $2H_0$.

For the dipole, the inclination, I (direction of the Earth's field away from the horizontal plane), at any θ is defined from

$$\tan(I) = \frac{Z}{H} = 2\cot(\theta). \tag{1.29}$$

This is a valuable relationship for measurements of continental drift (see Section 5.10). It means that we can determine our geomagnetic latitude ($90^\circ - \theta$) from field measurements of H and Z. Using ancient rocks to tell the field direction in an earlier geological time, the apparent latitude of the region can be fixed by Equation (1.29). Later, in Section 1.9, there will be more details regarding this paleomagnetism subject.

Conjugate points on the Earth's surface are locations P and P' that can be connected by a single dipole field line (Figure 1.8). The relatively strong Earth's field lines become guiding tracks for charged particles in the magnetosphere. The positions for conjugate points are used in studies of the Earth arrival of these phenomena from distant locations in space. The dipole field lines will extend out into the equatorial plane a distance, r_e. Up to about 65° geomagnetic latitude, θ' (in degrees), the length of this field line can be approximated by the relationship

$$\text{length} \approx 0.38\theta' r_e \tag{1.30}$$

where the length and r_e are in similar units (e.g., kilometers or Earth radii). Figure 1.9 shows the relationship of latitude and field-line equatorial distance. As an illustration, at 50° geomagnetic latitude, read the appropriate x-axis scale; move vertically to the curve intersection, then read horizontally to the corresponding y-axis scale, obtaining 2.5 Earth radii for the distant extent of that field line. We shall see, in Chapter 3, that the outermost field lines of the Earth's dipole field are distorted

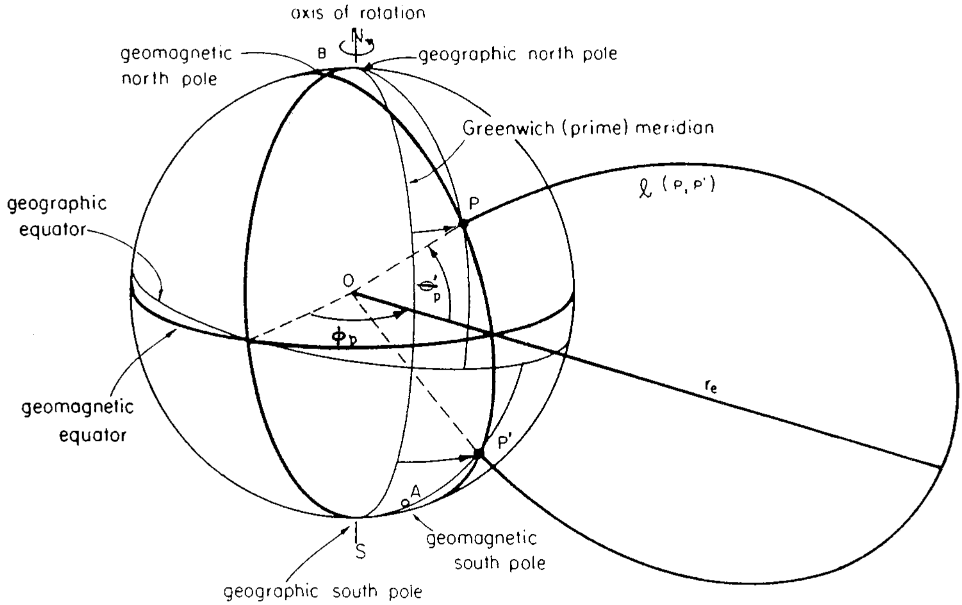


Figure 1.8. Geomagnetic locations, based on a spherical coordinate system aligned with respect to the dipole field, with latitude $\theta' = 90 - \theta$ (where θ is the colatitude) and longitude ϕ. A dipole field line of length l, connecting the conjugate points at P and P', extends to a distance r_e in the equatorial plane from the dipole center 0.

by a wind of particles and fields that arrive from the Sun; such change becomes quite noticeable above 60°.

The *magnetic shell parameter,* L shell, is an effective mean equatorial radius of a magnetic field shell, which, for a given field strength, B, defines the trapped-particle flux in the space about the Earth. Computation of the L shells for the Earth's field is complex. However, for a dipole field, the L-shell values may be considered almost equivalent to the number of Earth radii that the field line extends into space, r_e, and is a good approximation for all but the high latitudes. The *invariant latitude* (in degrees) of a location is obtained from L by the relationship

$$\cos(\text{invariant latitude}) = \frac{1}{\sqrt{L}} \tag{1.31}$$

Figure 1.10 shows polar views of the L-shell contours for the two hemispheres, computed for the model, extremely quiet field of 1965. Many of the high-latitude geomagnetic phenomena are best organized when plotted with respect to their L shell or invariant latitude positions.

1.4 Full Representation of the Main Field

Now comes the most difficult part of this book, the representation of the main field by equations and tables. There is a considerable amount

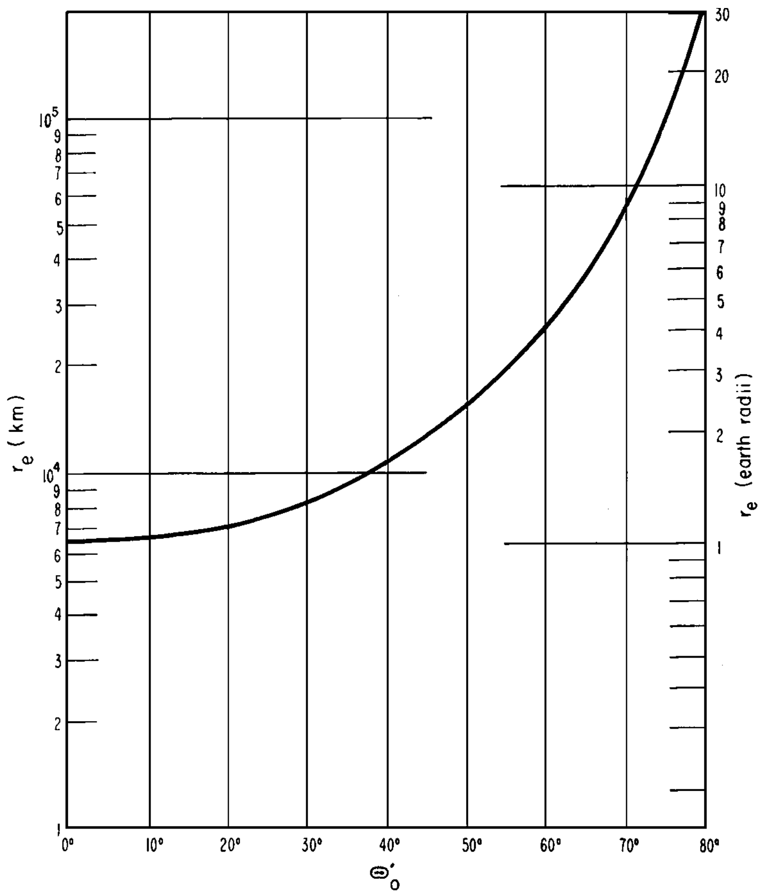
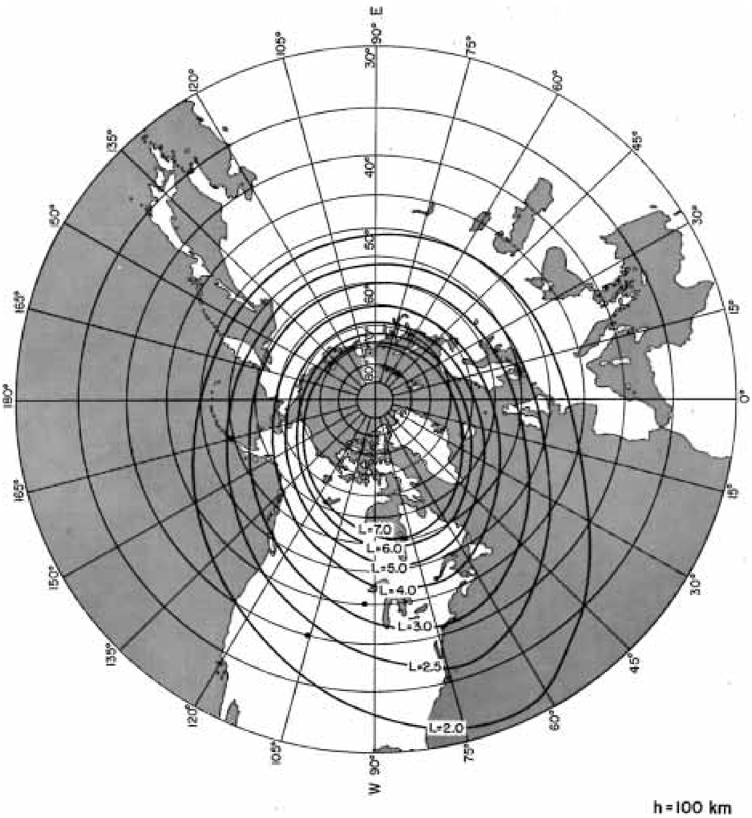


Figure 1.9. Equatorial radial extent r_e (from the Earth's center) of a dipole field line starting from latitude θ' at the Earth's surface.

of mathematics here, with analysis techniques and shorthand math symbols that can frighten the casual reader. I will try to step gently in this section, but it is necessary for us to go through the details because there will be so many important physical results we can properly appreciate later if we understand their origin in the main field representation.

We will start with some of Maxwell's equations and show how the relationships appear in a spherical coordinate system. Then we will look for a solution of the equations of a type that will let us separate current sources that arise above and below a sphere's surface. Next, we will look at a method for fitting the measurements from a surface of observatory field values into the equations that produce our Earth's field models. It is important to know some of the strengths and failings of the methods so that we understand their successful application in geophysics. We will also find the main field representation important in the chapter on



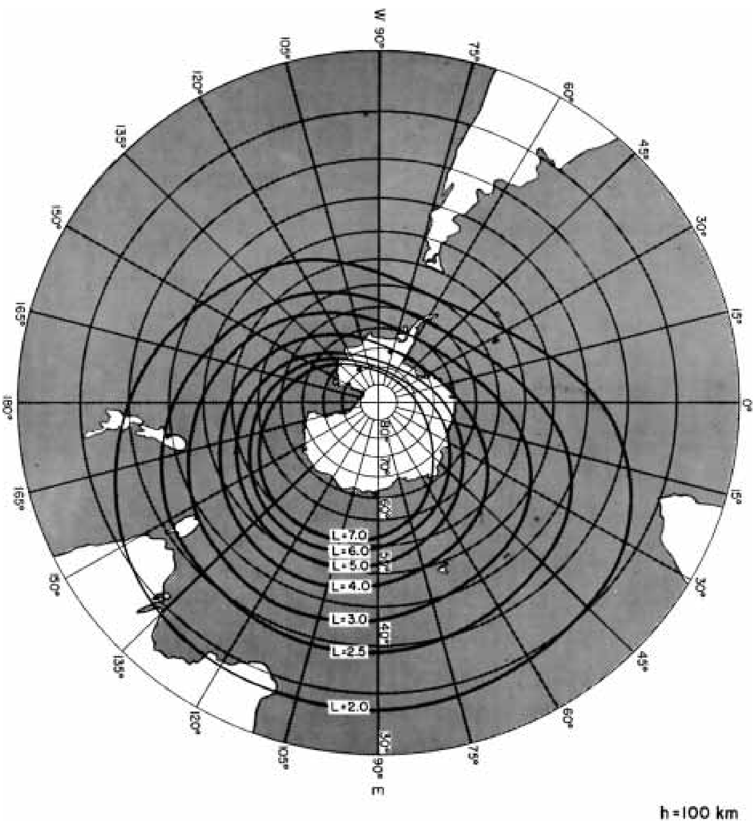


Figure 1.10. *L*-shell contours, computed for 100-km altitude, in the Northern (top) and Southern (bottom) Hemisphere regions. Geographic east and west radial longitude lines and circles of latitude (from 30° to the pole) are shown. These *L*-values were computed for the extremely quiet year, 1965, when there was a minimum distortion of the polar contours by solar wind.

quiet-field variations as well as in our discussion of solar–terrestrial disturbances. If you are not ready for the mathematics at this time, at least read through the steps lightly to focus upon what is being computed and the sequence followed.

Maxwell's great contribution to the understanding of electromagnetic phenomena was to show that all the measurements and laws of field behavior could be derived from a few compact mathematical expressions. We will start with one of these equations, adjusted for the assumptions that only negligible electric field changes occur and that the amount of current flowing across the boundary between the Earth and its atmosphere is relatively insignificant. Then, at the Earth's surface

$$\nabla \times \mathbf{B} = \mathbf{i}\left(\frac{\delta B_z}{\delta y} - \frac{\delta B_y}{\delta z}\right) + \mathbf{j}\left(\frac{\delta B_x}{\delta z} - \frac{\delta B_z}{\delta x}\right) + \mathbf{k}\left(\frac{\delta B_y}{\delta x} - \frac{\delta B_x}{\delta y}\right) = 0, \quad (1.32)$$

where **i, j, k** represent the three orthogonal directions and δ indicates that "partial" derivatives are used (see Section A.7). This equation is read "the curl of B equals zero" and requires that the field can be obtained from the "negative gradient of a scalar potential" so

$$\mathbf{B} = -\left[\mathbf{i}\frac{\delta V}{\delta x} + \mathbf{j}\frac{\delta V}{\delta y} + \mathbf{k}\frac{\delta V}{\delta z}\right] = -\nabla V. \quad (1.33)$$

The other Maxwell's equation that we will use is

$$\nabla \cdot \mathbf{B} = \left[\frac{\delta B_x}{\delta x} + \frac{\delta B_y}{\delta y} + \frac{\delta B_z}{\delta z}\right] = 0. \quad (1.34)$$

This equation is read "the divergence of the field is zero." Now, putting Equations (1.33) and (1.34) together, we obtain

$$\nabla \cdot \nabla V = \nabla^2 V = 0, \quad (1.35)$$

which is read as "the Laplacian of scalar V is zero." This *potential function* will be valid over a spherical surface through which current does not flow. In spherical coordinate notation, Equation (1.35) becomes

$$\frac{\delta}{\delta r}\left(r^2\frac{\delta V}{\delta r}\right) + \frac{1}{\sin\theta}\frac{\delta}{\delta\theta}\left(\sin\theta\frac{\delta V}{\delta\theta}\right) + \frac{1}{\sin^2\theta}\frac{\delta^2 V}{\delta\phi^2} = 0, \quad (1.36)$$

in which r, θ, and ϕ are the geographic, Earth-centered coordinates of the radial distance, colatitude, and longitude, respectively.

Now, the solution (i.e., solving the equation for an expression of V by itself) that is sought is one that is a product of three expressions. The first of these expressions is to be only a function of r; the second, only a function of θ; and the third, only a function of ϕ. That is what mathematicians call a "separable" solution of the form

$$V(r, \theta, \phi) = R(r) \cdot S(\theta, \phi), \quad \text{where } S(\theta, \phi) = T(\theta) \cdot L(\phi). \tag{1.37}$$

A solution of the potential function V for the Earth's main field satisfying these requirements has the converging series of terms (devised by Gauss in 1838)

$$V = a \sum_{n=1}^{\infty} \left[\left(\frac{r}{a}\right)^n S_n^e + \left(\frac{a}{r}\right)^{n+1} S_n^i \right], \tag{1.38}$$

where the $\sum$ means the sum of terms as n goes from 1 to an extremely large number, and for our studies, a is the Earth radius, R_e. The series solution means that for each value of n the electromagnetic laws are obeyed as if that term were the only contribution to the field. We will soon see that solving this equation for V allows us to immediately recover the strength of the magnetic field components at any location about the Earth.

There are two series for V. The first is made up of terms in r^n. As r increases, these terms become larger and larger; that means we must be approaching the current source of an external field in the increasing r direction. These terms are called V_e, "the external source terms of the potential function" (and our reason for labeling the S_n functions with a superscript e). By a corresponding argument for the second series, the $(1/r)^n$ terms become larger and larger as r becomes smaller and smaller, which means we must be approaching the current source of an internal field in the decreasing r direction. Scientists call these terms V_i, "the internal source terms of the potential function" (the reason for labeling the corresponding S_n functions with a superscript i).

The $S(\theta, \phi)$ terms of Equation (1.37) represent sets of a special class of functions called *Legendre polynomials* (see Section C.11) of the independent variable θ that are multiplied by sine and cosine function terms of independent variable ϕ. I shall leave to more detailed textbooks the explanation of what is called the required "orthogonality" properties and "normalization" and simply define the "Schmidt quasi-normalized, associated Legendre polynomial functions" that are used for global field analysis. Here I will abbreviate these as Legendre polynomials, $P_n^m(\theta)$, realizing they are a special subgroup of functions. The integers, n and m, are called *degree* and *order*, respectively; n has a value of 1 or greater, and m is always less than or equal to n.

When V is determined from measurements of the field about the Earth, analyses show that essentially all the contribution comes from the V_i part of the potential function expansion. For now, let us just call this

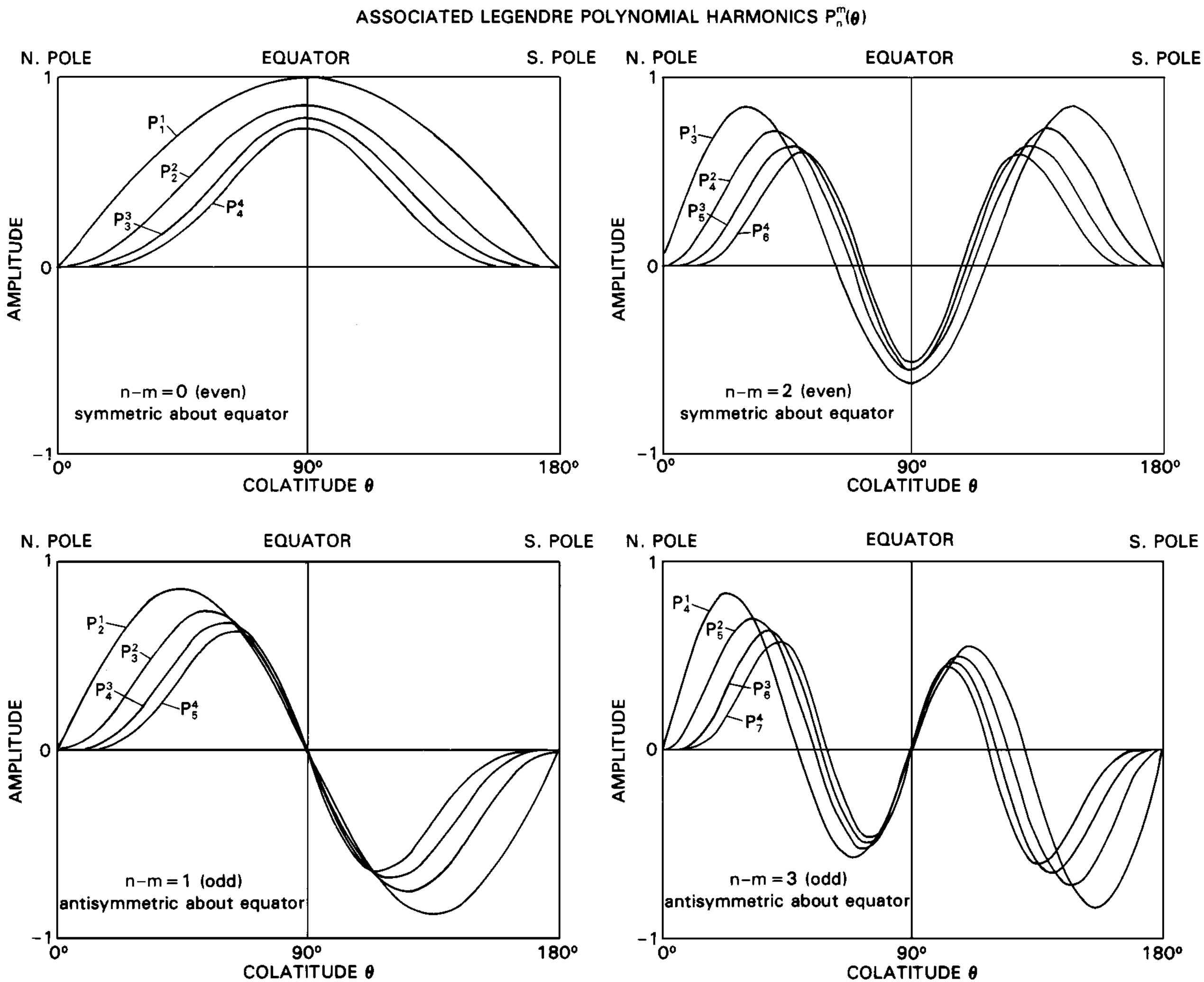


Figure 1.11. Examples of the associated Legendre polynomial, P_n^m, variations with colatitude, θ, from the North Pole (0°) through the equator (90°) to the South Pole (180°) for selected values of degree n and order m. The four sets are separated for similar values of $(n - m)$.