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1 Fundamentals of laser energy absorption

1.1 Classical electromagnetic-theory concepts

1.1.1 Electric and magnetic properties of materials

Electric and magnetic fields can exert forces directly on atoms or molecules, resulting in changes in the distribution of charges. Thus, an electric field $\vec{E}$ induces an electric dipole moment or polarization vector $\vec{P}$, while the magnetic induction field $\vec{B}$ drives a magnetic dipole moment or magnetization vector $\vec{M}$. It is convenient to define the electric displacement vector $\vec{D}$ and the magnetic field $\vec{H}$ such that

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \tag{1.1}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}, \tag{1.2}$$

where ε_0 and μ_0 are the electric permittivity and magnetic permeability, respectively, in vacuum. For isotropic electric materials the vectors $\vec{D}$, $\vec{E}$, and $\vec{P}$ are collinear, while correspondingly for isotropic magnetic materials the vectors $\vec{H}$, $\vec{B}$, and $\vec{M}$ are collinear.

Introducing the electric susceptibility χ, the polarization vector is written as

$$\vec{P} = \chi \varepsilon_0 \vec{E}, \tag{1.3}$$

and, therefore,

$$\vec{D} = \varepsilon_0(1+\chi)\vec{E} = \varepsilon_{\rm r}\varepsilon_0\vec{E} = \varepsilon\vec{E}, \tag{1.4}$$

where ε is the electric permittivity of the material and $\varepsilon_{\rm r} = \varepsilon/\varepsilon_0$ the relative permittivity. Analogous expressions are used to describe the magnetic properties of materials:

$$\vec{B} = \mu\vec{H} = \mu_{\rm r}\mu_0\vec{H}, \tag{1.5}$$

where μ is the material's magnetic permeability and $\mu_{\rm r}$ the relative magnetic permeability. In a medium where the charge density ρ moves with velocity $\vec{v}$, the free current-density vector $\vec{J}$ is defined as

$$\vec{J} = \rho\vec{v}. \tag{1.6}$$

The magnitude of this current, $|\vec{J}|$, represents the net amount of positive charge crossing a unit area normal to the instantaneous direction of $\vec{v}$ per unit time. The current-density vector is related to the electric field vector via the electric conductivity, σ,

$$\vec{J} = \sigma \vec{E}, \tag{1.7}$$

which is the continuum form of Ohm's law. In isotropic materials, σ is a scalar quantity, but for crystalline or anisotropic solids σ is a second-order tensor.

1.1.2 Maxwell's equations

The system of Maxwell's equations constitutes the basis for the theory of electromagnetic fields and waves as well as their interactions with materials. For macroscopically homogeneous (uniform) materials, for which ε and μ are constants independent of position, the following relations hold:

(I)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}, \tag{1.8}$$

(II)

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{1.9}$$

(III)

$$\nabla \cdot \vec{D} = \rho, \tag{1.10}$$

(IV)

$$\nabla \cdot \vec{B} = 0, \tag{1.11}$$

where $\vec{D}$, $\vec{B}$, and $\vec{J}$ are related to $\vec{E}$ and $\vec{H}$ through the constitutive Equations (1.4), (1.5), and (1.7). In vacuum where there is no current or electric charge, the Maxwell equations have a simple traveling plane wave solution with the electric and magnetic field orthogonal to one another, and to the direction of propagation.

1.1.2 Boundary conditions

Consider an interface i, separating two media (1) and (2) of different permittivities ε_1, ε_2 and permeabilities μ_1, μ_2 (Figure 1.1). According to Born and Wolf (1999) the sharp and distinct interface is replaced by an infinitesimally thin transition layer. Within this layer ε and μ are assumed to vary continuously. Let $\vec{n}_{12}$ be the local normal at the interface pointing into the medium (2). An elementary cylinder of volume δV and surface area δA is taken within the thin transition layer. The cylinder faces and peripheral wall are normal and parallel to vector $\vec{n}_{12}$, respectively. Since $\vec{B}$ and its derivatives may be assumed continuous over this elementary control volume, the Gauss divergence theorem

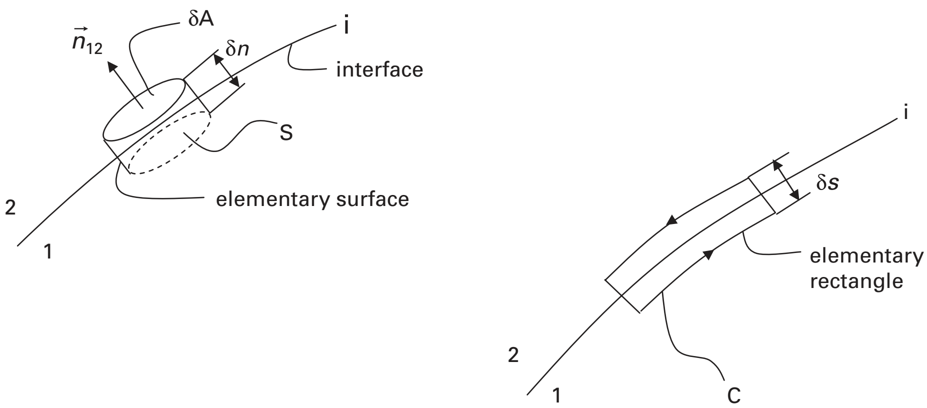

Figure 1.1. Schematics of an elementary volume of height δn and an elementary rectangular contour of width δs across the distinct interface separating media 1 and 2.

is applied to Equation (1.11):

$$\int\int_{\delta V}\int \nabla\cdot\vec{B}\,\mathrm{d}V = \int_{\delta S}\int \vec{B}\cdot \mathrm{d}\vec{S} = 0. \tag{1.12}$$

The second integral is taken over the surface of the cylinder. In the limit, as the height of the cylinder $\delta h \to 0$, contributions from the peripheral wall vanish and this integral yields

$$(\vec{B}_1\cdot\vec{n}_1 + \vec{B}_2\cdot\vec{n}_2)\delta A = 0, \tag{1.13}$$

where $\vec{n}_1 = -\vec{n}_{12}$ and $\vec{n}_2 = \vec{n}_{12}$. Consequently,

$$\vec{n}_{12}\cdot(\vec{B}_2 - \vec{B}_1) = 0. \tag{1.14}$$

The electric displacement vector $\vec{D}$ is treated in a similar manner by applying the Gauss theorem to Equation (1.10):

$$\int\int_{\delta V}\int \nabla\cdot\vec{D}\,\mathrm{d}V = \int_{\delta S}\int \vec{D}\cdot\mathrm{d}\vec{S} = \int\int_{\delta V}\int \rho\,\mathrm{d}V. \tag{1.15}$$

In the limit,

$$\lim_{\delta h\to 0}\int\int_{\delta V}\int \rho\,\mathrm{d}V = \int_{\delta A}\int \sigma_{\mathrm{s}}\,\mathrm{d}A. \tag{1.16}$$

The above relation defines the surface charge density σ_{s}. Owing to the vanishing contribution over the peripheral wall as $\delta h \to 0$, Equation (1.16) gives

$$\vec{n}_{12}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma_{\mathrm{s}}. \tag{1.17}$$

These boundary conditions (1.14) and (1.17) can be expressed as

$$B_{2\mathrm{n}} = B_{1\mathrm{n}}, \tag{1.18a}$$

$$D_{2\mathrm{n}} - D_{1\mathrm{n}} = \sigma_{\mathrm{s}}, \tag{1.18b}$$

where $B_{2n} = \vec{B}_2 \cdot \vec{n}$, $B_{1n} = \vec{B}_1 \cdot \vec{n}$, $D_{2n} = \vec{D}_2 \cdot \vec{n}$, and $D_{1n} = \vec{D}_1 \cdot \vec{n}$. In other words, the normal components of the magnetic induction vector $\vec{B}$ are always continuous and the difference between the normal components of the electric displacement $\vec{D}$ is equal in magnitude to the surface charge density σ_s.

To examine the behavior of the tangential electric and magnetic field components, a rectangular contour C with two long sides parallel to the surface of discontinuity is considered. Stokes' theorem is applied to Equation (1.8):

$$\int\int_{\delta S} \nabla \times \vec{E} \cdot d\vec{S} = \int\int_{\delta S} \nabla \times \vec{E} \cdot \vec{s}\, dS = \int_{C} \vec{E} \cdot d\vec{l} = -\mu \int\int_{\delta S} \frac{\partial \vec{H}}{\partial t} \cdot \vec{s}\, dS. \tag{1.19}$$

In the limit as the width of the rectangle $\delta h \to 0$, the last surface integral vanishes and the contour integral of $\vec{E}$ is reduced to

$$\vec{E}_1 \cdot \vec{t}_1 + \vec{E}_2 \cdot \vec{t}_2 = 0. \tag{1.20}$$

Considering the unit tangent vector $\vec{t}$ along the interface, $\vec{t}_1 = -\vec{t} = -\vec{s} \times \vec{n}_{12}$, $\vec{t}_2 = \vec{t} = \vec{s} \times \vec{n}_{12}$, Equation (1.20) gives

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0. \tag{1.21}$$

If a similar procedure is applied to Equation (1.9), then

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}, \tag{1.22}$$

where $\vec{K}$ is the surface current density.

The boundary conditions (1.21) and (1.22) are written in the following form:

$$E_{2t} = E_{1t}, \tag{1.23a}$$

$$H_{2t} - H_{1t} = K_t. \tag{1.23b}$$

The subscript t implies the tangential component of the field vector. Thus, the tangential component of the electric field vector $\vec{E}$ is always continuous at the boundary surface and the difference between the tangential components of the magnetic vector $\vec{H}$ is equal to the line current density K, and in radiation problems where $\sigma_s = 0$, $K = 0$. Consequently, the normal components of $\vec{D}$ and $\vec{B}$ and the tangential components of $\vec{E}$ and $\vec{H}$ are continuous across interfaces separating media of different permittivities and permeabilities.

1.1.3 Energy density and energy flux

Light carries energy in the form of electromagnetic radiation. For a single charge q_e, the rate of work done by an external electric field $\vec{E}$ is $q_e\vec{v} \cdot \vec{E}$, where $\vec{v}$ is the velocity of the charge. If there exists a continuous distribution of charge and current, the total rate of work per unit volume is $\vec{J} \cdot \vec{E}$, since $\vec{J} = \rho\vec{v}$. Utilizing (1.9),

$$\vec{J} \cdot \vec{E} = \vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}. \tag{1.24}$$

The following identity is invoked:

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}), \tag{1.25}$$

and applied to (1.24):

$$\vec{J} \cdot \vec{E} = -\nabla \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}. \tag{1.26}$$

The above equation is cast as follows:

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{\boldsymbol{S}} = -\vec{J} \cdot \vec{E}, \tag{1.27a}$$

where

$$U = \frac{1}{2}(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}), \tag{1.27b}$$

$$\vec{\boldsymbol{S}} = \vec{E} \times \vec{H}. \tag{1.27c}$$

The scalar U represents the energy density of the electromagnetic field and in the SI system has units of [J/m^3]. The vector $\vec{\boldsymbol{S}}$ is called the Poynting vector and has units [W/m^2]. It is consistent to view $|\vec{S}|$ as the power per unit area transported by the electromagnetic field in the direction of $\vec{\boldsymbol{S}}$. Hence, the quantity $\nabla \cdot \vec{\boldsymbol{S}}$ quantifies the net electromagnetic power flowing out of a unit control volume. Equation (1.27a) states the *Poynting vector theorem*.

1.1.4 Wave equations

Recalling the vector identity $\nabla \times (\nabla \times) = \nabla \cdot (\nabla \cdot) - \nabla^2$, Equation (1.8) yields

$$\nabla \times \nabla \times \vec{E} = \nabla \cdot (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t}. \tag{1.28}$$

Invoking (1.9), the right-hand side of the above is

$$-\mu \nabla \times \frac{\partial \vec{H}}{\partial t} = -\mu \frac{\partial \vec{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2},$$

and (1.28) gives

$$\nabla^2 \vec{E} - \nabla\left(\frac{\rho}{\varepsilon}\right) = \mu\left(\frac{\partial \vec{J}}{\partial t} + \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}\right),$$

or

$$\nabla^2 \vec{E} - \nabla\left(\frac{\rho}{\varepsilon}\right) = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{1.29a}$$

Similarly, it can be shown that

$$\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}. \tag{1.29b}$$

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For propagation in vacuum, $\rho = 0$, $\sigma = 0$, $\mu = \mu_0$, and $\varepsilon = \varepsilon_0$, and Equations (1.29a) and (1.29b) give

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \tag{1.30a}$$

$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}. \tag{1.30b}$$

The above are wave equations indicating a speed of wave propagation $c_0 = 1/\sqrt{\mu_0 \varepsilon_0}$, i.e. the speed of light in vacuum. For propagation in a perfect dielectric, $\rho = 0$, $\sigma = 0$, and the following apply

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}, \tag{1.31a}$$

$$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}. \tag{1.31b}$$

The propagation speed in this case is $c = 1/\sqrt{\mu\varepsilon}$. The index of refraction then is defined:

$$n = \frac{c_0}{c} = \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}. \tag{1.32}$$

Since, at optical frequencies, $\mu_0 \cong \mu$, the refractive index is approximated as

$$n \cong \sqrt{\frac{\varepsilon}{\varepsilon_0}}. \tag{1.33}$$

Equations (1.30) and (1.31) can be satisfied by monochromatic plane-wave solutions with a constant amplitude A and of the general form

$$\psi = A \mathrm{e}^{\mathrm{i}(\omega t - \vec{r} \cdot \vec{s})}, \tag{1.34}$$

where $\vec{r}$ and $\vec{s}$ are the position vector and the wavevector, respectively.

The angular frequency ω and the magnitude of the wavevector $\vec{s}$ are related by

$$|\vec{s}| = \omega \sqrt{\mu\varepsilon}. \tag{1.35}$$

According to (1.34), the field has the same values at locations $\vec{r}$ and times t that satisfy

$$\omega t - \vec{r} \cdot \vec{s} = \text{const}. \tag{1.36}$$

The above prescribes a plane normal to the wavevector $\vec{s}$ at any time instant t (Figure 1.2). The plane is called a *surface of constant phase*, often referred to as a *wavefront*.

The plane-wave electromagnetic fields are expressed by

$$\vec{E} = \vec{u}_1 E_0 \mathrm{e}^{\mathrm{i}(\omega t - \vec{r} \cdot \vec{s})}, \tag{1.37a}$$

$$\vec{H} = \vec{u}_2 H_0 \mathrm{e}^{\mathrm{i}(\omega t - \vec{r} \cdot \vec{s})}, \tag{1.37b}$$

where $\vec{u}_1$ and $\vec{u}_2$ are constant unit vectors and E_0 and H_0 are the constant-in-space complex amplitudes.

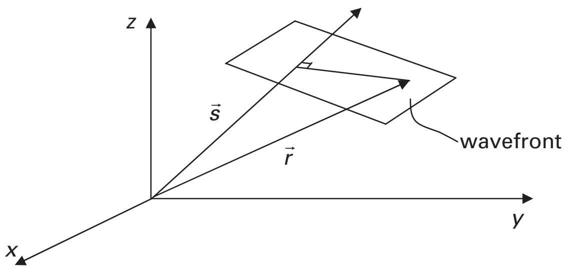

Figure 1.2. A schematic diagram depicting a plane wave propagating normal to the direction $\vec{s}$.

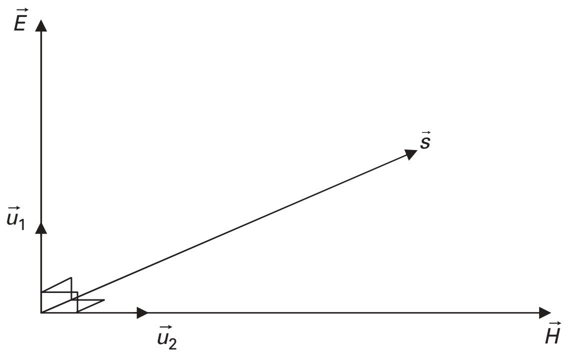

Figure 1.3. A schematic diagram depicting the instantaneous vectors $\vec{E}$ and $\vec{H}$ that form a right-hand triad with the unit vector $\vec{s}$ along the propagation direction.

In a homogeneous, charge-free medium, $\nabla \cdot \vec{E} = \nabla \cdot \vec{H} = 0$. Hence,

$$\vec{u}_1 \cdot \vec{s} = \vec{u}_2 \cdot \vec{s} = 0, \tag{1.38}$$

meaning that $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation (Figure 1.3). For this reason, electromagnetic waves in dielectrics are said to be *transverse*.

The curl Maxwell equations impose further restrictions on the field vectors. By applying (1.38) in (1.8), it can be shown that

$$\vec{u}_2 = \frac{\vec{s} \times \vec{u}_1}{|\vec{s}|}. \tag{1.39}$$

The triad $(\vec{u}_1, \vec{u}_2, \vec{s})$ therefore forms a set of orthogonal vectors, and $\vec{E}$ and $\vec{H}$ are in phase with amplitudes in constant ratio, provided that ε and μ are both real (Figure 1.4). The plane wave described is a transverse wave propagating in the direction $\vec{s}$ with a time-averaged flux of energy

$$\vec{S} = \frac{|E_0|^2}{2\omega\mu}\vec{u}_3 = \frac{\vec{E}^* \cdot \vec{E}}{2\omega\mu}\vec{u}_3, \tag{1.40}$$

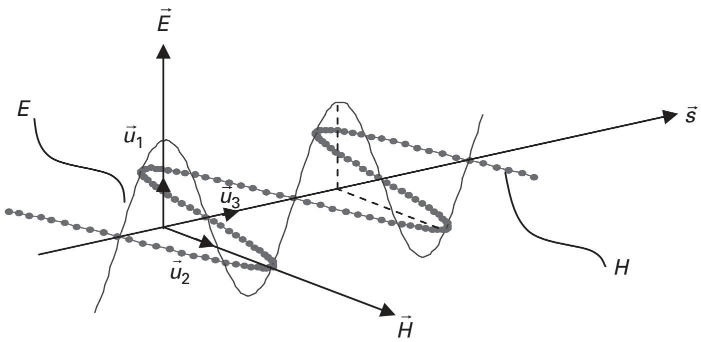

Figure 1.4. In a medium of real refractive index, the electric and magnetic fields are always in phase.

where $\vec{E}^*$ is the conjugate of the complex electric field vector. The time-averaged energy density

$$U = \frac{1}{2}\varepsilon|E_0|^2 = \frac{1}{2}\varepsilon\vec{E}\cdot\vec{E}^*. \tag{1.41}$$

1.1.5 Electromagnetic theory of absorptive materials

The optical properties of perfect dielectric media are completely characterized by the real refractive index. In such media, it is assumed that electromagnetic radiation interacts with the constituent atoms with no energy absorption. In contrast, especially for metals, very little light penetrates to a depth beyond 1 μm at visible wavelengths. Consider then media with nonzero electric conductivity that absorb energy but do not redirect a collimated light beam. Let $\vec{E}$ and $\vec{H}$ be the real parts of periodic variations:

$$\vec{E}(x, y, z, t) = \mathrm{Re}[\vec{E}^{\mathrm{c}}(x, y, z)\mathrm{e}^{\mathrm{i}\omega t}], \tag{1.42a}$$

$$\vec{H}(x, y, z, t) = \mathrm{Re}[\vec{H}^{\mathrm{c}}(x, y, z)\mathrm{e}^{\mathrm{i}\omega t}]. \tag{1.42b}$$

The superscript c indicates a complex quantity. Utilizing Maxwell's equations (1.8)–(1.11),

$$\nabla\times\vec{E}^{\mathrm{c}} = -\mathrm{i}\omega\mu\vec{H}^{\mathrm{c}}, \tag{1.43a}$$

$$\nabla\times\vec{H}^{\mathrm{c}} = (\sigma+\mathrm{i}\omega\varepsilon)\vec{E}^{\mathrm{c}}, \tag{1.43b}$$

$$\nabla\cdot\vec{E}^{\mathrm{c}} = 0, \tag{1.43c}$$

$$\nabla\cdot\vec{H}^{\mathrm{c}} = 0. \tag{1.43d}$$

Taking the curl of (1.43a) and combining this with (1.43b) gives

$$\nabla\times\nabla\times\vec{E}^{\mathrm{c}} = -\mathrm{i}\omega\mu(\sigma+\mathrm{i}\omega\varepsilon)\vec{E}^{\mathrm{c}}. \tag{1.44}$$

Utilizing the identity $\nabla\times\nabla\times\vec{E}^{\mathrm{c}} = \nabla\cdot(\nabla\cdot\vec{E}^{\mathrm{c}}) - \nabla^2\vec{E}^{\mathrm{c}}$ combined with (1.43c),

$$\nabla\times\nabla\times\vec{E}^{\mathrm{c}} = -\nabla^2\vec{E}^{\mathrm{c}}. \tag{1.45}$$

Combining the above with (1.43a) and (1.43b) gives

$$\nabla^2 \vec{E}^{\mathrm{c}} = \mathrm{i}\omega\mu(\sigma + \mathrm{i}\omega\varepsilon)\vec{E}^{\mathrm{c}}, \tag{1.46}$$

or

$$\nabla^2 \vec{E}^{\mathrm{c}} = -\omega^2\mu\left(\varepsilon - \mathrm{i}\frac{\sigma}{\omega}\right)\vec{E}^{\mathrm{c}}, \tag{1.47}$$

which is written as

$$\nabla^2 \vec{E}^{\mathrm{c}} + (k^{\mathrm{c}})^2 \vec{E}^{c} = 0. \tag{1.48}$$

The complex wavenumber k^{c} satisfies

$$(k^{\mathrm{c}})^2 = \omega^2\mu\left(\varepsilon - \mathrm{i}\frac{\sigma}{\varpi}\right) = \omega^2\mu\varepsilon^{\mathrm{c}}.$$

The quantity

$$\varepsilon^{\mathrm{c}} = \varepsilon - \mathrm{i}\frac{\sigma}{\omega}$$

is the complex dielectric constant.

A complex velocity v^{c} and a complex refractive index n^{c} can then be defined:

$$v^{\mathrm{c}} = \frac{1}{\sqrt{\mu\varepsilon^{\mathrm{c}}}}, \tag{1.49}$$

$$n^{\mathrm{c}} = \frac{c_0}{v^{\mathrm{c}}} = \sqrt{\frac{\mu\varepsilon^{\mathrm{c}}}{\mu_0\varepsilon_0}}. \tag{1.50}$$

Let $n^{\mathrm{c}} = n - \mathrm{i}k$, where n is the real part of the complex refractive index and k the imaginary part, the so-called attenuation index:

$$(n^{\mathrm{c}})^2 = n^2 - k^2 - 2\mathrm{i}nk = \mu c_0^2\left(\varepsilon - \mathrm{i}\frac{\sigma}{\omega}\right). \tag{1.51}$$

Equating the real and the imaginary parts, and solving for n^2 and k^2, gives

$$n^2 = \frac{c_0^2}{2}\left[\sqrt{\mu^2\varepsilon^2 + \left(\frac{\mu\sigma}{2\pi\nu}\right)^2} + \mu\varepsilon\right], \tag{1.52a}$$

$$n^2 = \frac{c_0^2}{2}\left[\sqrt{\mu^2\varepsilon^2 + \left(\frac{\mu\sigma}{2\pi\nu}\right)^2} - \mu\varepsilon\right]. \tag{1.52b}$$

Equation (1.48) implies wave propagation. The simplest solution is that of a plane, time-harmonic wave:

$$\vec{E}^{\mathrm{c}}(\vec{r}, t) = \vec{E}_0^{\mathrm{c}} \mathrm{e}^{-\mathrm{i}[k^{\mathrm{c}}(\vec{r}\cdot\vec{s})-\omega t]}, \tag{1.53}$$

where $\vec{s}$ is a unit vector along the direction of propagation. Since

$$k^{\mathrm{c}} = \frac{\omega n^{\mathrm{c}}}{c_0} = \frac{\omega(n - \mathrm{i}k)}{c_0},$$

the above can be written as

$$\vec{E}^{c} = \vec{E}_0^{c} e^{-\frac{\omega}{c_0}k(\vec{r}\cdot\vec{s})} e^{i\omega[-\frac{n}{c_0}(\vec{r}\cdot\vec{s})+t]}. \tag{1.54}$$

The real part of this expression represents the electric vector:

$$\vec{E} = \vec{E}_0 e^{-\frac{\omega}{c_0}k(\vec{r}\cdot\vec{s})} \cos\left\{\omega\left[-\frac{n}{c_0}(\vec{r}\cdot\vec{s})+t\right]\right\}. \tag{1.55}$$

A similar expression can be developed for the magnetic field vector. The energy flux per unit area is given by the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, which is then

$$\vec{S} = \mathrm{Re}\left(\vec{E}_0^{c} e^{i\omega t}\right) \times \mathrm{Re}\left(\vec{H}_0^{c} e^{i\omega t}\right), \tag{1.56}$$

and then

$$\vec{S} = \frac{1}{4}\left[\left(\vec{E}_0^{c} e^{i\omega t} + \vec{E}_0^{c^*} e^{-i\omega t}\right)\right] \times \left[\left(\vec{H}_0^{c} e^{i\omega t} + \vec{H}_0^{c^*} e^{-i\omega t}\right)\right],$$
$$\vec{S} = \frac{1}{4}\left[\vec{E}_0^{c} \times \vec{H}_0^{c} e^{2i\omega t} + \vec{E}_0^{c^*} \times \vec{H}_0^{c^*} e^{-2i\omega t} + \vec{E}_0^{c^*} \times \vec{H}_0^{c} + \vec{E}_0^{c} \times \vec{H}_0^{c^*}\right]. \tag{1.57}$$

Consider a time interval $[-T', T']$ large compared with the fundamental wave period, $T = 2\pi/\omega$, which is $O(10^{-15}\ \mathrm{s})$:

$$\frac{1}{2T'}\int_{-T'}^{T'} e^{2i\omega t}\,dt = \frac{1}{4i\omega T'}\left[e^{2i\omega t}\right]_{-T'}^{T'} = \frac{1}{4i\frac{2\pi}{T}T'} 2\cos(\omega T') = \frac{T}{2\pi i T'}\cos(\omega T'). \tag{1.58}$$

Evidently, if the EM energy flux given in (1.57) is averaged over $[-T', T']$ with $T' \gg T$, the first two periodic terms will contribute very little. Hence, the averaged energy is

$$\vec{S}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E}_0^{c} \times \vec{H}_0^{c^*}\right) = \vec{s}\,\frac{1}{2}\frac{\mathrm{Re}(n^c)}{\mu c_0}\left|\vec{E}_0^{c}\right|^2. \tag{1.59}$$

The above expressions indicate that the energy flux carried by a wave propagating in an absorbing medium is proportional to the squared modulus of its complex amplitude and to the real part of the complex refractive index of the medium. The modulus of the Poynting vector, i.e. the monochromatic radiative intensity, I'_λ, is

$$I'_\lambda = |\vec{S}_{av}| = I'_{\lambda,0} e^{-\frac{2\omega}{c_0}k(\vec{r}\cdot\vec{s})} = I'_{\lambda,0} e^{-\gamma(\vec{r}\cdot\vec{s})}, \tag{1.60}$$

where γ is the absorption coefficient of the medium:

$$\gamma = \frac{2\omega k}{c_0} = \frac{4\pi k}{\lambda_0}. \tag{1.61}$$

In the above, λ_0 is the wavelength in vacuum. As shown in Figure 1.5, the energy flux drops to 1/e of $I'_{\lambda,0}$ after traveling a distance d, the so-called absorption penetration depth:

$$d = \frac{1}{\gamma} = \frac{\lambda_0}{4\pi k}. \tag{1.62}$$