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0521819989 - Generic Polynomials: Constructive Aspects of the Inverse Galois Problem

Christian U. Jensen, Arne Ledet and Noriko Yui

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This book describes a constructive approach to the inverse Galois problem: Given a finite group  $G$  and a field  $K$ , determine whether there exists a Galois extension of  $K$  whose Galois group is isomorphic to  $G$ . Further, if there is such a Galois extension, find an explicit polynomial over  $K$  whose Galois group is the prescribed group  $G$ .

The main theme of the book is an exposition of a family of “generic” polynomials for certain finite groups, which give all Galois extensions having the required group as their Galois group. The existence of such generic polynomials is discussed, and where they do exist, a detailed treatment of their construction is given. The book also introduces the notion of “generic dimension” to address the problem of the smallest number of parameters required by a generic polynomial.

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**Generic Polynomials**  
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## Contents

Acknowledgments	ix
Introduction	1
0.1. The Inverse Problem of Galois Theory	1
0.2. Milestones in Inverse Galois Theory	3
0.3. The Noether Problem and Its History	5
0.4. Strategies	8
0.5. Description of Each Chapter	9
0.6. Notations and Conventions	13
0.7. Other Methods	15
Chapter 1. Preliminaries	17
1.1. Linear Representations and Generic Polynomials	17
1.2. Resolvent Polynomials	23
Exercises	26
Chapter 2. Groups of Small Degree	29
2.1. Groups of Degree 3	30
2.2. Groups of Degree 4	31
2.3. Groups of Degree 5	38
2.4. Groups of Degree 6	50
2.5. Groups of Degree 7	51
2.6. Groups of Degree 8, 9 and 10	56
2.7. Groups of Degree 11	57
Exercises	60
Chapter 3. Hilbertian Fields	63
3.1. Definition and Basic Results	63
3.2. The Hilbert Irreducibility Theorem	67
3.3. Noether's Problem and Dedekind's Theorem	71
Exercises	80
Chapter 4. Galois Theory of Commutative Rings	83
4.1. Ring Theoretic Preliminaries	83
4.2. Galois Extensions of Commutative Rings	84
4.3. Galois Algebras	90
Exercises	93

Chapter 5. Generic Extensions and Generic Polynomials	95
5.1. Definition and Basic Results	95
5.2. Retract-Rational Field Extensions	98
5.3. Cyclic Groups of Odd Order	102
5.4. Regular Cyclic 2-Extensions and Ikeda's Theorem	106
5.5. Dihedral Groups	109
5.6. $p$ -Groups in characteristic $p$	117
Exercises	123
Chapter 6. Solvable Groups I: $p$ -Groups	127
6.1. Quaternion Groups	128
6.2. The Central Product $QC$	142
6.3. The Quasi-Dihedral Group	146
6.4. The Cyclic Group of Order 8	152
6.5. The Dihedral Group $D_8$	155
6.6. Heisenberg Groups	161
Exercises	165
Chapter 7. Solvable Groups II: Frobenius Groups	169
7.1. Preliminaries	169
7.2. Wreath Products and Semi-Direct Products	173
7.3. Frobenius Groups	175
Exercises	180
Chapter 8. The Number of Parameters	187
8.1. Basic Results	187
8.2. Essential Dimension	190
8.3. Lattices: Better Bounds	196
8.4. $p$ -Groups in Characteristic $p$ , Revisited	201
8.5. Generic Dimension	201
Exercises	204
Appendix A. Technical Results	207
A.1. The 'Seen One, Seen Them All' Lemma	207
A.2. Tensor Products	210
A.3. Linear Disjointness	213
A.4. The Hilbert Nullstellensatz	214
Appendix B. Invariant Theory	217
B.1. Basic Concepts	217
B.2. Invariants	220
B.3. Bracket Polynomials	222
B.4. The First Fundamental Theorem of Invariant Theory	227
Exercises	244
Bibliography	247
Index	255

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