Part I

Classical mechanics

The visible Universe contains hundreds of billions of galaxies, each consisting of billions of stars. Recent discoveries of extrasolar planets lead us to believe that a typical galaxy may contain billions of planets (and presumably, asteroids and comets). The planets, stars and galaxies interact on a hierarchy of scales ranging from AU to parsec to megaparsec, experiencing forces arising from gravity, on all scales, and cosmic expansion on the larger scales. The combination of gravitational attraction and cosmic expansion has shaped the visible matter in the Universe into a hierarchy of structures leading to clusters and superclusters of galaxies.

A full description of the interactions that define the large-scale structure of the Universe and its constituent parts requires the application of general relativity on all scales and the introduction of a new force, as embodied in the recently proposed cosmological constant, on the largest scales. In this part, however, we limit ourselves largely to the application of classical (Newtonian) mechanics which is sufficiently accurate to describe the topics covered in this part and has the advantage of being more intuitive and accessible to the reader.

This part begins with a review of the basic elements of classical mechanics, subsequently used to derive Kepler's laws, the Virial theorem and various aspects of orbital motion. The resulting derivations are applied to specific astrophysical problems such as planetary motion, extrasolar planets, binary stars, galaxy rotation curves, dark matter, the large scale structure of the Universe and cosmic expansion.

Chapter 1

Orbital mechanics

I begin this part by reviewing some basic concepts that underlie Newtonian gravitation. The concepts of universal gravitation, center of mass and reduced mass are defined and subsequently used in the following chapters.

1.1 Universal gravitation

The gravitational force acting between two bodies, m_1 and m_2, located at $\vec{R}_1$ and $\vec{R}_2$, is given by

$$\vec{F} = \pm \frac{G m_1 m_2}{|\vec{R}_1 - \vec{R}_2|^3} (\vec{R}_1 - \vec{R}_2) \tag{1}$$

where the quantities are defined in Fig. 1.1 and G is the gravitational constant. The $\pm$ signs reflect the fact that the same magnitude of force acts on m_1 and m_2 but with opposite sign.

1.1.1 Center of mass

Consider a point on a line, joining m_1 and m_2, which is the centroid of the total mass distribution. We call this centroid the *center of mass* of the two-body system. The vector $\vec{r}$, separating the two masses, can then be decomposed into $\vec{r}_1$ and $\vec{r}_2$ relative to the center of mass, such that

$$\vec{r} = \vec{r}_1 - \vec{r}_2.$$

From Newton's Second Law

$$\vec{F}_1 = m_1 \ddot{\vec{r}}_1 = -\frac{G m_1 m_2}{|\vec{r}|^3} \vec{r}. \tag{2}$$

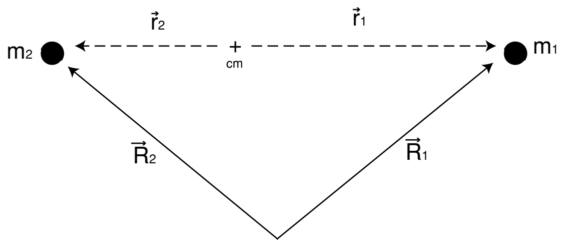

Fig. 1.1 The universal law of gravitation. Gravity is a mutual force that acts between the masses m_1 and m_2.

Similarly

$$\vec{F}_2 = \frac{Gm_1m_2}{|\vec{r}|^3}\vec{r} \tag{3}$$

$$\Rightarrow \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \ddot{\vec{r}} = -\frac{G(m_1+m_2)}{|\vec{r}|^3}\vec{r} = -\frac{GM}{|\vec{r}|^3}\vec{r}. \tag{4}$$

The acceleration of the two bodies toward each other is proportional to the total mass and inversely proportional to the square of the distance between them. The location of the center of mass (CM) can now be found

$$m_1\ddot{\vec{r}}_1 = -m_2\ddot{\vec{r}}_2 \Rightarrow -m_1\frac{GM}{|\vec{r}|^3}\vec{r}_1 = m_2\frac{GM}{|\vec{r}|^3}\vec{r}_2 \Rightarrow \vec{r}_1 = -\frac{m_2}{m_1}\vec{r}_2 \tag{5}$$

where r_1 and r_2 represent the distance of m_1 and m_2 from the center of mass, respectively. The center of mass is a useful concept in astronomy. It marks the center about which two astronomical bodies orbit. In an isolated two-body system, the center of mass is not seen to accelerate.

1.1.2 Reduced mass

Let us define a mass such that

$$\vec{F} = \mu\ddot{\vec{r}} = -\frac{GM\mu}{|\vec{r}|^3}\vec{r} = -\frac{Gm_1m_2}{|\vec{r}|^3}\vec{r}$$

$$\Rightarrow \mu = \frac{m_1m_2}{m_1+m_2}. \tag{6}$$

The concept of reduced mass allows us to transform any two-body problem into a one-body problem where the reduced mass responds to a central force emanating from a point whose distance is equal to the separation of the original two bodies.

1.2 Kepler's laws

We are now in a position to derive the most famous orbital laws used in astronomy, Kepler's laws. We begin, as with so many other problems in classical mechanics, with the Lagrangian

$$L = T - V \tag{7}$$

where T is the kinetic energy and V is the potential energy. Let us set $m = m_1$ and $M = m_2$ in anticipation of defining planetary orbits where the planets have much lower masses than the Sun (that is $m \ll M$). We are considering a two-body interaction so that the expected motion is in a plane and possibly periodic. It therefore makes sense to use polar coordinates, r and θ for this problem. Equation (7) then becomes

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{8}$$

We are now in a position to determine the angular momentum p_θ from the Lagrangian. Recall that

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

We now use the Lagrange equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0.$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0.$$

Integrating

$$mr^2\dot{\theta} = l = \text{constant}. \tag{9}$$

Equation (9) represents the *conservation of angular momentum*. Rearranging terms

$$\frac{1}{2}r^2\dot{\theta} = \frac{1}{2}\frac{l}{m} = \text{constant}.$$

Recall that the area of an elemental triangle is given by $\mathrm{d}A = r^2/2\ \mathrm{d}\theta$, so that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{r^2}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \text{constant}. \tag{10}$$

According to (10), a radius vector sweeps out equal areas in equal time which, of course, is *Kepler's Second Law*.

The Hamiltonian or total energy of a two-body system is given by

$$E = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r). \tag{11}$$

Rearranging (11) and solving for $\dot{r}$

$$\dot{r}^2 = \frac{2}{m}(E - V(r)) - r^2\dot{\theta}^2.$$

But, $r^2\dot{\theta} = l/m$ so that

$$\dot{r}^2 = \frac{2}{m}(E - V(r)) - \left(\frac{l}{rm}\right)^2$$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}.$$

The above can be solved for dt so that

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{(2/m)(E - V(r) - (l^2/(2mr^2)))}}. \tag{12}$$

Equation (12) can now be used to determine the shape of the orbit resulting from the two-body interaction. What we really want is a function $r(\theta)$ which means converting (12) into a relationship between r and θ and eliminating t in the process.

We begin by noting that $r^2\dot{\theta} = r^2(\mathrm{d}\theta/\mathrm{d}t) = l/m$ so that $l\,\mathrm{d}t = mr^2\,\mathrm{d}\theta$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} = \frac{l}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

so that

$$\frac{mr^2}{l}\,\mathrm{d}\theta = \frac{\mathrm{d}r}{\sqrt{(2/m)(E - V(r) - (l^2/(2mr^2)))}}$$

$$\Rightarrow \mathrm{d}\theta = \frac{l\,\mathrm{d}r}{mr^2\sqrt{(2/m)(E - V(r) - (l^2/(2mr^2)))}}$$

$$\Rightarrow \theta = \int_{r_0}^{r} \frac{\mathrm{d}r}{r^2\sqrt{(2mE/l^2) - (2mV/l^2) - (1/r^2)}} + \theta_0. \tag{13}$$

Let $\mu = 1/r$ and substitute into (13)

$$\Rightarrow \theta = \theta_0 - \int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu}{\sqrt{(2mE/l^2) - (2mV/l^2) - \mu^2}}.$$

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For $V = -(GmM)/r = -k/r = -k\mu$

$$\Rightarrow \theta = \theta_0 - \int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu}{\sqrt{(2mE/l^2) + (2mk\mu/l^2) - \mu^2}} \tag{14}$$

which can be put into standard form and solution with $\mu = x$

$$\int \frac{\mathrm{d}x}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{-c}} \cos^{-1}\left[\frac{-b + 2cx}{q}\right] \tag{15}$$

where

$$q = b^2 - 4ac.$$

Comparison of (14) and (15) yields

$$a = \frac{2mE}{l^2} \qquad b = \frac{2mk}{l^2} \qquad c = -1$$
$$q = \left(\frac{2mk}{l^2}\right)^2 \left(1 + \frac{2El^2}{mk^2}\right)$$

so that the solution to (14) becomes

$$\theta = \theta' - \cos^{-1}\left[\frac{(l^2\mu/mk) - 1}{\sqrt{1 + (2El^2/mk^2)}}\right] \tag{16}$$

where θ' incorporates the additional constants resulting from the integration. Putting $\mu = 1/r$ back into (16) and taking the cosine of both sides yields

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\theta - \theta')\right). \tag{17}$$

We now have a solution, $r(\theta)$, that determines the shape of the orbit and clearly depends on the energy, E, and the angular momentum, l. This equation can be compared with the general expression for a conic section

$$\frac{1}{r} = C(1 + \epsilon \cos(\theta - \theta')). \tag{18}$$

By equating (17) to (18) we see that

$$C = \frac{mk}{l^2} \tag{19}$$
$$\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}.$$

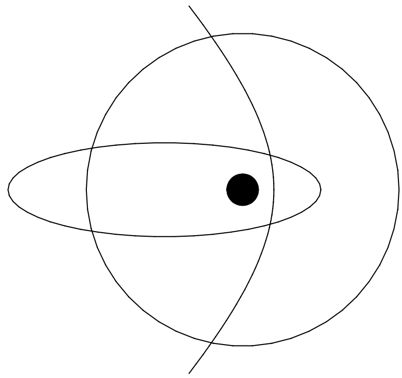

Fig. 1.2 Orbits as conic sections. Circular, elliptical and parabolic/hyperbolic classes of orbits are shown. The total energy, E, determines the class of orbit while the combination of E and the angular momentum, l, determines the shape of the orbit within a class. The Sun is shown as the small filled circle at the center.

The only variable that can be negative is the total energy of the two-body system so that

$$E > 0 \rightarrow \epsilon > 1 \qquad \text{hyperbola}$$
$$E = 0 \rightarrow \epsilon = 1 \qquad \text{parabola}$$
$$E < 0 \rightarrow \epsilon < 1 \qquad \text{ellipse}$$
$$E = -\frac{1}{2}V = -\frac{mk^2}{2l^2} \rightarrow \epsilon = 0 \qquad \text{circle.}$$

These define conic sections, as illustrated in Fig. 1.2.

In the solar system, planets have closed orbits ($E < 0$) and move in elliptical trajectories (*Kepler's First Law*). Kepler's Third Law can now be derived, beginning with the second law. Integrating (10) over a complete period of the orbit yields

$$\int_0^P \dot{A}\, \mathrm{d}t = \frac{1}{2}\frac{l}{m}P = \pi ab \tag{20}$$

where πab is the area of an ellipse and a and b are the semi-major and semi-minor axes of the elliptical orbit. Now from (18) we can define a as the sum of distances that correspond to $\theta = \theta'$ and $\theta = \theta' + \pi$

$$a = \frac{1}{C(1-\epsilon^2)}.$$

Combining this with the well-known relationship between a and b

$$b = a\sqrt{(1-\epsilon^2)}$$

yields

$$b = \sqrt{\frac{a}{C}}. \tag{21}$$

Combining (19) and (21) yields

$$b = \sqrt{a}\sqrt{\frac{l^2}{mk}}. \tag{22}$$

Combining (20) and (22)

$$\begin{aligned} \frac{1}{2}\frac{l}{m}P &= \pi a^{3/2}\sqrt{\frac{l^2}{mk}} \\ \Rightarrow P &= 2\pi a^{3/2}\sqrt{\frac{m}{k}} \\ \Rightarrow P &= \frac{2\pi}{\sqrt{GM}}\, a^{3/2}. \end{aligned} \tag{23}$$

Equation (23) represents *Kepler's Third Law* – the square of the period is proportional to the cube of the diameter of the orbit.

1.2.1 Planetary orbits

The planets follow orbits as described by (18). However, the orbits differ significantly from each other and do not fall in exactly the same plane. Consequently, it is necessary to describe planetary orbits in three dimensions relative to a standard reference frame, as shown in Fig. 1.3.

There are two major reference points for a planetary orbit and both are related to the Earth. The Earth's orbit (plane NB) is used as the standard reference plane called the *ecliptic*. The intersection of the Earth's celestial equator with the ecliptic defines the *vernal* and *autumnal equinoxes*. The former is denoted as γ in Fig. 1.3. It is used as the fundamental reference point for defining the orbital elements. The plane of the Earth's orbit (the ecliptic) is γN′B while the plane of the planet's orbit is NQN′. The intersection of the two planes is called the *line of nodes* which connect the *ascending* and *descending nodes* (N and N′, respectively – the direction of motion of the planet is indicated by the arrow). The Sun is located at the center and its position is denoted by S. The true orbit of the planet is shown as the ellipse pLA. The *perihelion* position is marked as A and the position of the planet, at time t, is denoted as p. The planet and the Sun define a radius vector, Sp, that cuts the great circle, NQN′, at P1.

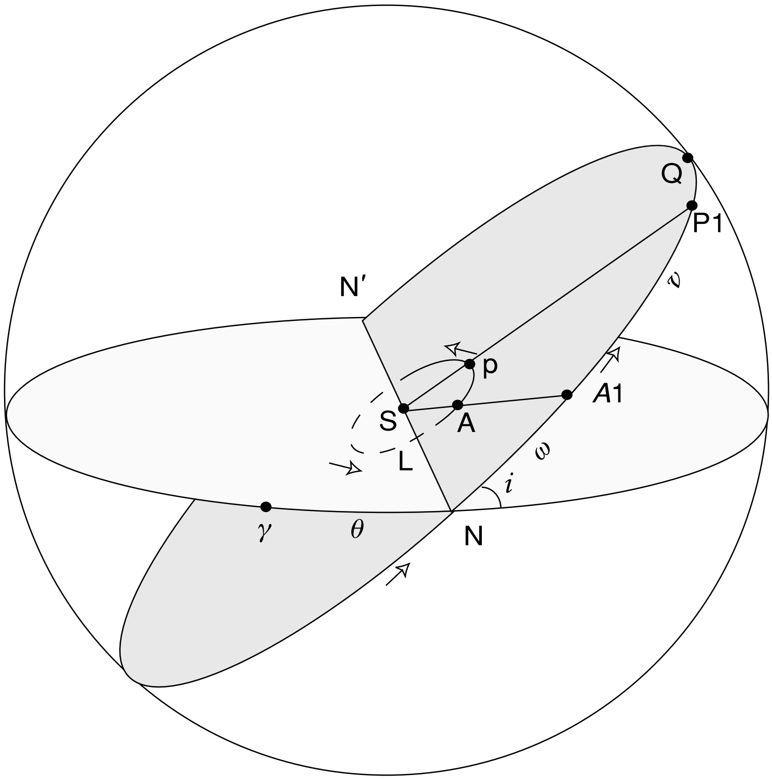

Fig. 1.3 The orbit of a planet relative to the Earth's orbit. A planetary orbit can be uniquely defined in 3-D space relative to γ and the Earth's orbit. The various parameters that characterize the planetary orbit are defined in the text.

With the help of Fig. 1.3, we can define the following parameters of the *apparent* orbit of the planet

$$v = A1 - P1 = \text{true anomaly}$$
$$\omega = \text{N} - A1 = \text{argument of perihelion}$$
$$\theta = \gamma - \text{N} = \text{longitude of ascending node}$$
$$\bar{\omega} = \theta + \omega = \text{longitude of the perihelion}$$
$$\text{L} = \theta + \omega + v = \text{true longitude of planet}$$
$$i = \text{B} - \text{N} - A1 = \text{inclination of orbit}$$
$$\tau = \text{time when planet is at perihelion, A.}$$

The six elements that completely define the orbit are $a, e, \theta, \bar{\omega}, i, \tau$. To complete the connection to (18), which we derived earlier, we see that $v = \theta - \theta'$. The

Table 1.1. *Planetary orbits – elements on January 1, 2000*

Planet	a (AU)	e	P (years)	i (degree)	θ (degree)	$\theta + \omega$ (degree)
Mercury	0.387	0.206	0.241	7.00	48.33	77.46
Venus	0.723	0.007	0.615	3.39	76.68	131.53
Earth	1.000	0.017	1.000	0.0	−11.26	102.95
Mars	1.524	0.093	1.85	1.85	49.58	336.04
Jupiter	5.203	0.048	11.862	1.31	100.56	14.75
Saturn	9.537	0.054	29.458	2.49	113.72	92.43
Uranus	19.191	0.047	84.012	0.77	74.23	170.96
Neptune	30.069	0.009	164.796	1.77	131.72	44.97
Pluto	39.482	0.249	246.378	17.14	110.30	224.07

additional elements allow us to determine the orbit relative to our perspective at the Earth. Table 1.1 lists the orbital elements of the planets in our solar system.

1.3 Binary stars

1.3.1 Visual binaries

Roughly half of all stars in the Galaxy are binaries. Analysis of binary star orbits via the equations we have derived thus far, provides valuable information regarding stellar properties and stellar evolution, information that would otherwise be difficult to obtain. Binary systems in which both stars are visible are known as visual binaries.

1.3.2 The apparent orbit

Binary stars represent the most general two-body problem. Their orbits are oriented randomly in space and are described fully in three dimensions in much the same way as were the planets we discussed earlier. However, because we only see a projection of the orbit on the sky we must somehow recover the orbital elements from an analysis of the 2-D orbit. The 2-D orbit is measured according to Fig. 1.4.

The most general form of an ellipse is given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + 1 = 0 \tag{24}$$

where $x = \rho \cos\theta$ and $y = \rho \sin\theta$ and all coefficients are real constants. The equation of the apparent orbit is obtained by fitting (24) to a large number of measurements of ρ and θ. The more observations the better the fit and the more accurate the coefficients that define the shape of the apparent orbit. The procedures for