

Cambridge University Press  
0521818338 - Combinatorics of Symmetric Designs  
Yury J. Ionin and Mohan S. Shrikhande  
Frontmatter  
[More information](#)

---

## **Combinatorics of Symmetric Designs**

The aim of this book is to provide a unified exposition of the theory of symmetric designs with emphasis on recent developments. The authors cover the combinatorial aspects of the theory giving particular attention to the construction of symmetric designs and related objects. The last five chapters of the book are devoted to balanced generalized weighing matrices, decomposable symmetric designs, subdesigns of symmetric designs, non-embeddable quasi-residual designs, and Ryser designs. Most results in these chapters have never previously appeared in book form. The book concludes with a comprehensive bibliography of over 400 entries.

Researchers in all areas of combinatorial designs, including coding theory and finite geometries, will find much of interest here. Detailed proofs and a large number of exercises make this book suitable as a text for an advanced course in combinatorial designs.

YURY J. IONIN is a professor of mathematics at Central Michigan University, USA.

MOHAN S. SHRIKHANDE is a professor of mathematics at Central Michigan University, USA.

Cambridge University Press  
0521818338 - Combinatorics of Symmetric Designs  
Yury J. Ionin and Mohan S. Shrikhande  
Frontmatter  
[More information](#)

---

New Mathematical Monographs

**Editorial Board**

Béla Bollobás, *University of Memphis*

William Fulton, *University of Michigan*

Frances Kirwan, *Mathematical Institute, University of Oxford*

Peter Sarnak, *Princeton University*

Barry Simon, *California Institute of Technology*

For information about Cambridge University Press mathematics publications visit  
<http://www.cambridge.org/mathematics>

Cambridge University Press  
0521818338 - Combinatorics of Symmetric Designs  
Yury J. Ionin and Mohan S. Shrikhande  
Frontmatter  
[More information](#)

---

# Combinatorics of Symmetric Designs

YURY J. IONIN and MOHAN S. SHRIKHANDE  
*Central Michigan University*



**CAMBRIDGE**  
UNIVERSITY PRESS

Cambridge University Press  
0521818338 - Combinatorics of Symmetric Designs  
Yury J. Ionin and Mohan S. Shrikhande  
Frontmatter  
[More information](#)

---

CAMBRIDGE UNIVERSITY PRESS  
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo  
Cambridge University Press  
The Edinburgh Building, Cambridge CB2 2RU, UK  
Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9780521818339](http://www.cambridge.org/9780521818339)

© Cambridge University Press 2006

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without  
the written permission of Cambridge University Press.

First published 2006

Printed in the United Kingdom at the University Press, Cambridge

*A catalog record for this publication is available from the British Library*

*Library of Congress Cataloging in Publication data*

ISBN-13 978-0-521-81833-9 hardback  
ISBN-10 0-521-81833-8 hardback

---

Cambridge University Press has no responsibility for the persistence or accuracy of URLs  
for external or third-party internet websites referred to in this publication, and does not  
guarantee that any content on such websites is, or will remain, accurate or appropriate.

---

Cambridge University Press  
0521818338 - Combinatorics of Symmetric Designs  
Yury J. Ionin and Mohan S. Shrikhande  
Frontmatter  
[More information](#)

---

*To Irina, Tania, and Timur*

*To Neelima, Aditi, and Sean*

## Contents

---

<i>Preface</i>	<i>page xi</i>
<b>1 Combinatorics of finite sets</b>	<b>1</b>
1.1 Fisher's Inequality	1
1.2 The First Ray-Chaudhuri–Wilson Inequality	3
1.3 Symmetric designs and Ryser designs	5
1.4 Equidistant families of sets	8
Exercises	11
Notes	12
<b>2 Introduction to designs</b>	<b>14</b>
2.1 Incidence structures	14
2.2 Graphs	19
2.3 Basic properties of $(v, b, r, k, \lambda)$ -designs	24
2.4 Symmetric designs	28
2.5 The Bruck–Ryser–Chowla Theorem	34
2.6 Automorphisms of symmetric designs	38
2.7 A symmetric $(41, 16, 6)$ -design	42
2.8 A symmetric $(79, 13, 2)$ -design	48
Exercises	53
Notes	56
<b>3 Vector spaces over finite fields</b>	<b>59</b>
3.1 Finite fields	59
3.2 Affine planes and nets	61
3.3 The 36 officers problem	66
3.4 Projective planes	72
3.5 Affine geometries over finite fields	76

viii	<i>Contents</i>	
	3.6 Projective geometries over finite fields	79
	3.7 Combinatorial characterization of $PG_{n-1}(n, q)$	86
	3.8 Two infinite families of symmetric designs	95
	3.9 Linear codes	97
	Exercises	103
	Notes	110
<b>4</b>	<b>Hadamard matrices</b>	113
	4.1 Basic properties of Hadamard matrices	113
	4.2 Kronecker product constructions	116
	4.3 Conference matrices	118
	4.4 Regular Hadamard matrices	126
	4.5 From Paley matrices to regular Hadamard matrices	132
	4.6 Regular sets of $(\pm 1)$ -matrices	133
	4.7 Binary equidistant codes	144
	Exercises	150
	Notes	152
<b>5</b>	<b>Resolvable designs</b>	154
	5.1 Bose's Inequality	154
	5.2 Affine $\alpha$ -resolvable designs	161
	5.3 Resolvable 2-designs	163
	5.4 Embedding of resolvable designs in symmetric designs	172
	5.5 Resolvable 2-designs and equidistant codes	182
	Exercises	184
	Notes	184
<b>6</b>	<b>Symmetric designs and <math>t</math>-designs</b>	186
	6.1 Basic properties of $t$ -designs	186
	6.2 The Second Ray-Chaudhuri–Wilson Inequality	191
	6.3 Hadamard 3-designs	193
	6.4 Cameron's Theorem	195
	6.5 Golay codes and Witt designs	198
	6.6 Symmetric designs with parameters $(56, 11, 2)$ and $(176, 50, 14)$	203
	Exercises	207
	Notes	210
<b>7</b>	<b>Symmetric designs and regular graphs</b>	212
	7.1 Strongly regular graphs	212
	7.2 Eigenvalues of strongly regular graphs	219

*Contents*

ix

7.3	Switching in strongly regular graphs	223
7.4	Symmetric designs with polarities	233
7.5	Symmetric designs and digraphs	239
	Exercises	243
	Notes	245
<b>8</b>	<b>Block intersection structure of designs</b>	247
8.1	Association schemes	247
8.2	Quasi-symmetric designs	250
8.3	Multiples of symmetric designs	259
8.4	Quasi-3 symmetric designs	263
8.5	Block schematic designs with three intersection numbers	270
8.6	Designs with a nearly affine decomposition	276
8.7	A symmetric $(71, 15, 3)$ -design	280
	Exercises	286
	Notes	286
<b>9</b>	<b>Difference sets</b>	289
9.1	Group invariant matrices and group rings	289
9.2	Singer and Paley–Hadamard difference sets	299
9.3	Symmetries in a group ring	301
9.4	Building blocks and building sets	307
9.5	McFarland, Spence, and Davis–Jedwab difference sets	310
9.6	Relative difference sets	313
	Exercises	319
	Notes	321
<b>10</b>	<b>Balanced generalized weighing matrices</b>	323
10.1	Basic properties of BGW-matrices	323
10.2	BGW-matrices with classical parameters	331
10.3	BGW-matrices and relative difference sets	336
10.4	Kronecker product constructions	341
10.5	BGW-matrices and projective geometries	354
	Exercises	365
	Notes	366
<b>11</b>	<b>Decomposable symmetric designs</b>	368
11.1	A symmetric $(66, 26, 10)$ -design	368
11.2	Global decomposition of symmetric designs	369
11.3	Six infinite families of globally decomposable symmetric designs	374



11.4	Productive Hadamard matrices	376
11.5	Symmetric designs with irregular global decomposition	383
11.6	Decomposable symmetric designs and regular graphs	386
11.7	Local decomposition of symmetric designs	391
11.8	Infinite families of locally decomposable symmetric designs	397
11.9	An infinite family of designs with a nearly affine decomposition	402
	Exercises	406
	Notes	406
<b>12</b>	<b>Subdesigns of symmetric designs</b>	<b>407</b>
12.1	Tight subdesigns	407
12.2	Examples of tight subdesigns	412
12.3	Normal subdesigns	421
12.4	Symmetric designs with $M$ -arcs	424
	Exercises	427
	Notes	427
<b>13</b>	<b>Non-embeddable quasi-residual designs</b>	<b>429</b>
13.1	Quasi-residuals of non-existing symmetric designs	429
13.2	Linear non-embeddability conditions	431
13.3	BGW-matrices and non-embeddability	436
13.4	Non-embeddable quasi-derived designs	443
	Exercises	445
	Notes	446
<b>14</b>	<b>Ryser designs</b>	<b>447</b>
14.1	Basic properties of Ryser designs	447
14.2	Type-1 Ryser designs	456
14.3	Ryser designs of prime index	464
14.4	Ryser designs of small index	467
14.5	Ryser designs of small gcd	475
	Exercises	486
	Notes	486
	<i>Appendix</i>	488
	<i>References</i>	495
	<i>Index</i>	514

## Preface

---

Design theory is a well-established branch of combinatorial mathematics. The origins of the subject can be traced back to statistics in the pioneering works of R. A. Fisher, F. Yates, and R. C. Bose. From the very beginning, one of the central objects of design theory has been symmetric designs. The prototype of a symmetric design is a finite projective plane, and the theory of symmetric designs borrows its methods and ideas from finite geometries, group theory, number theory, and linear algebra.

It is notoriously difficult to construct an infinite family of symmetric designs or even a single symmetric design. However, in recent years new ideas in constructing symmetric designs have been discovered and new infinite families have been found. The central role in these constructions is played by balanced generalized weighing matrices. These matrices generalize the notion of a symmetric design but until recently they were often regarded as a rather obscure combinatorial object. Now they seem to be a useful tool in unifying different construction methods that have been developed since the 1950s.

This book is primarily a research monograph which aims to give a unifying exposition of the theory of symmetric designs with emphasis on these new developments. The book covers the combinatorial aspects of the theory with particular attention to constructing symmetric designs and related objects. Recent results that have never previously appeared in book format are developed mainly in the last five chapters. These chapters are devoted to balanced generalized weighing matrices, decomposable symmetric designs, subdesigns of symmetric designs, non-embeddable quasi-residual designs, and Ryser designs. The preceding chapters on finite geometries, Hadamard matrices, resolvable designs,  $t$ -designs, strongly regular graphs, and difference sets emphasize relations between these objects and symmetric designs.

We believe that this book can also be used as a text for a course in combinatorial designs. We begin with a brief introduction to combinatorial set theory,

including such beautiful results as Fisher's Inequality, the Ray-Chaudhuri–Wilson Inequality, and the Ryser–Woodall Theorem. The proofs of these theorems are elementary, but we hope they may be of interest even to the expert. Both Fisher's Inequality and the Ryser–Woodall Theorem allow us to introduce the notion of a symmetric design even before the formal definition is given in Chapter 2. Chapters 2–4 and 6–9 contain basic material on combinatorial designs, finite geometries, Hadamard matrices, strongly regular graphs, difference sets, and codes. We have included many examples and exercises and presented the proofs of many theorems in a manner suitable for graduate and advanced undergraduate students. Every chapter of the book is concluded by notes containing comments, references, and historical material. We suggest that the following chapters and sections could form a course in combinatorial designs: Chapter 1, Chapter 2 (without Sections 2.7 and 2.8), Chapter 3 (without Section 3.7), and also Sections 4.1, 4.2, 4.3, 6.1, 6.2, 6.3, 6.5, 7.1, 7.2, 9.1, and 9.2. A standard course of linear algebra and the basic notions of combinatorics and abstract algebra should form a sufficient background for this book.

The numbering of theorems, definitions, remarks, and examples is consecutive within each section and includes the chapter and section numbers, so, for instance, Theorem 3.7.10 can be found in Section 3.7. However, equations are numbered consecutively within each chapter. The last two sections of every chapter are **Exercises** and **Notes**. The **Appendix** contains the list of parameters of all known symmetric designs, which are combined into 23 series and 12 sporadic designs. We conclude the book with an extensive **References** section of over 400 entries, all of which are cited in the book.

We would like to acknowledge people and institutions who through their help, financial support, and hospitality made this work possible. Our particular thanks are due to Alphonse Baartmans, Dieter Jungnickel, Hadi Kharaghani, Vassili Mavron, Gary McGuire, Damaraju Raghavarao, Dijen Ray-Chaudhuri, S. S. Shrikhande, and Vladimir Tonchev for their comments and encouragement during various stages of preparation of this book.

We thank O. Abu Ghnaim, T. Al-Raqqad, J. R. Angelos, T. Ionin, D. Levi, A. Sarker, and K. W. Smith for help and comments and also the students of three classes at Central Michigan University who had to use imperfect drafts of the book as their textbooks.

Our own research that is included in this book, and the writing of the book were done at Central Michigan University, with extensive use of its facilities. The university has also supported us with sabbaticals and numerous travel grants. We are especially thankful to Central Michigan University for two Research Professorship grants awarded to each of us. We would also like to acknowledge the hospitality and financial support of the following

Cambridge University Press  
0521818338 - Combinatorics of Symmetric Designs  
Yury J. Ionin and Mohan S. Shrikhande  
Frontmatter  
[More information](#)

---

*Preface*

xiii

institutions: Mathematisches Forschungsinstitut, Oberwolfach, Germany; Michigan Technological University, Houghton, Michigan, USA; Ohio State University, Columbus, Ohio, USA; University of Lethbridge, Lethbridge, Alberta, Canada; Temple University, Philadelphia, Pennsylvania, USA; University of Wales, Aberystwyth, Wales, UK.

We thank Roger Astley and the staff of Cambridge University Press for their superb assistance during preparation and production of this book.

Finally, we would like to thank our wives for their unwavering support, patience, and understanding.