1

Combinatorics of finite sets

A number of advances in combinatorics originated in the following problem: given a finite set and a property of families of subsets of this set, estimate the size of a family with this property and then explore families of maximum or minimum size.

In this chapter we will discuss three problems of this kind:

(i) given a nonempty finite set V, estimate the size of a family $\mathcal{F}$ of subsets of V such that $|A \cap B|$ is the same for all distinct $A, B \in \mathcal{F}$;
(ii) given a nonempty finite set V and positive integers k and s, estimate the size of a family $\mathcal{F}$ of k-subsets of V such that $|A \cap B|$ takes at most s values for distinct $A, B \in \mathcal{F}$;
(iii) given a nonempty finite set V, estimate the size of a family $\mathcal{F}$ of subsets of V such that the cardinality of the symmetric difference of A and B is the same for all distinct $A, B \in \mathcal{F}$.

This discussion will lead us to *symmetric designs*, the central object of study in this book.

1.1. Fisher's Inequality

When we consider families of subsets of a finite set V of cardinality v, it is convenient to think of V as the set $\{1, 2, \ldots, v\}$ and associate with every subset X of V a $(0, 1)$-string $(x_1, x_2, \ldots, x_v)$ of length v where $x_i = 1$ if $i \in X$ and $x_i = 0$ if $i \notin X$.

We now introduce a simple but useful idea. In order to estimate the size of a family $\mathcal{F}$ of subsets of V, we will select a suitable finite-dimensional vector space P over the rationals and associate an element of P with each element of

$\mathcal{F}$. If the set of vectors associated with the elements of $\mathcal{F}$ is linearly independent, then the cardinality of $\mathcal{F}$ does not exceed the dimension of P.

As the first application of this idea, we take P to be the vector space of linear polynomials $a_0 + a_1x_1 + a_2x_2 + \cdots + a_vx_v$ in v variables with rational coefficients. Clearly, $\dim P = v + 1$. We will now give a proof of the following result:

Theorem 1.1.1 (Nonuniform Fisher's Inequality). *Let V be a nonempty finite set and $\mathcal{F}$ a family of subsets of V such that the cardinality of the intersection of any two distinct members of $\mathcal{F}$ is the same positive integer. Then $|\mathcal{F}| \le |V|$.*

Proof. Let $\mathcal{F}$ be a family of subsets of the set $V = \{1, 2, \ldots, v\}$. Assume there exists a positive integer λ such that $|A \cap B| = \lambda$ for any distinct A and B in $\mathcal{F}$.

Suppose first that there exists $A \in \mathcal{F}$ such that $|A| \le \lambda$. Then $|A| = \lambda$ and the intersection of any two distinct members of $\mathcal{F}$ is the set A. By subtracting A from each member of $\mathcal{F}$, we obtain a family of pairwise disjoint subsets of the set $V \setminus A$. Since the cardinality of such a family does not exceed $|V \setminus A| + 1$, we obtain that $|\mathcal{F}| \le v - \lambda + 1 \le v = |V|$.

From now on, we assume that $|A| > \lambda$ for any $A \in \mathcal{F}$. With each $A \in \mathcal{F}$, we associate the linear polynomial $f_A = \sum_{i \in A} x_i - \lambda$. Then $f_A(X) = |A \cap X| - \lambda$ for any $X \subseteq V$ (regarded as a (0, 1)-string). In particular, for any $A, B \in \mathcal{F}$,

$$f_A(B) = \begin{cases} 0 & \text{if } B \ne A, \\ |B| - \lambda & \text{if } B = A. \end{cases} \tag{1.1}$$

We claim that the subset $\{f_A : A \in \mathcal{F}\}$ of the vector space P is linearly independent. Indeed, if $\sum_{A \in \mathcal{F}} \alpha_A f_A = 0$ for some (rational) coefficients α_A, then, applying both sides of this equation to an arbitrary $B \in \mathcal{F}$ and using (1.1), we obtain that $\alpha_B(|B| - \lambda) = 0$, so $\alpha_B = 0$.

Suppose that the constant polynomial 1 is spanned by the polynomials f_A, $A \in \mathcal{F}$, i.e.,

$$1 = \sum_{A \in \mathcal{F}} \alpha_A f_A. \tag{1.2}$$

for some coefficients α_A. Then, applying both sides of (1.2) to $B \in \mathcal{F}$ and using (1.1), we obtain that $\alpha_B(|B| - \lambda) = 1$, so

$$1 = \sum_{A \in \mathcal{F}} \frac{1}{|A| - \lambda} f_A.$$

Applying both sides of this equation to the empty set, we obtain

$$1 = \sum_{A \in \mathcal{F}} \frac{-\lambda}{|A| - \lambda},$$

a contradiction, since the right-hand side of the last equation is negative.

Thus, the set $\{f_A : A \in \mathcal{F}\} \cup \{1\}$ of linear polynomials is linearly independent. Since $\dim P = v + 1$, we obtain that $|\mathcal{F}| + 1 \leq v + 1$, i.e., $|\mathcal{F}| \leq v = |V|$. □

The bound given by Fisher's Inequality is sharp. If $\mathcal{F}$ is the family of all $(v-1)$-subsets of the v-set V, then $|A \cap B| = v - 2$ for all distinct $A, B \in \mathcal{F}$ and $|\mathcal{F}| = v$.

1.2. The First Ray-Chaudhuri–Wilson Inequality

If A and B are distinct elements of a family $\mathcal{F}$ of subsets of a set V, the number $|A \cap B|$ is called an *intersection number* of $\mathcal{F}$. In the previous section, we considered families of subsets with one intersection number. In this section, we will consider families with s intersection numbers. To estimate the size of such a family, we will use the vector space P_s of multilinear polynomials of total degree s or less in v variables.

Definition 1.2.1. Let Q_s be the vector space of all polynomials in variables $x_1, x_2, \ldots, x_v$ of total degree $\leq s$ with rational coefficients. For each $I \subseteq \{1, 2, \ldots, v\}$, let $x_I = \prod_{i \in I} x_i$ (with the convention that $x_\varnothing = 1$). A polynomial $f \in Q_s$ is called *multilinear* if it can be represented as a linear combination of the polynomials x_I with $|I| \leq s$. For every polynomial f in variables $x_1, x_2, \ldots, x_v$, let f^* be the multilinear polynomial obtained by replacing each occurrence of x_i^k by x_i (for $k \geq 2$ and $i = 1, 2, \ldots, v$).

Multilinear polynomials form a subspace P_s of Q_s, and the polynomials x_I with $|I| \leq s$ form a basis of P_s. Therefore, $\dim P_s = \sum_{i=0}^{s} \binom{v}{i}$.

With every subset X of $\{1, 2, \ldots, v\}$, we again associate a $(0,1)$-string $(x_1, x_2, \ldots, x_v)$ of length v where $x_i = 1$ if $i \in X$ and $x_i = 0$ if $i \notin X$. Then, for any polynomial f in v variables, we have $f(X) = f^*(X)$.

Theorem 1.2.2 (The First Ray-Chaudhuri–Wilson Inequality). *Let $\mathcal{F}$ be a family of subsets of a set V of cardinality v. Let M be a set of non-negative integers, $|M| = s$. Suppose that $|A| = k$ is the same for all $A \in \mathcal{F}$, $|A \cap B| \in M$ for any distinct $A, B \in \mathcal{F}$, and $k > m$ for all $m \in M$. Then $|\mathcal{F}| \leq \binom{v}{s}$.*

Proof. Let $V = \{1, 2, \ldots, v\}$ and let $\mathcal{F}$ be a family of k-subsets of V satisfying the conditions of the theorem. With each $A \in \mathcal{F}$, we associate the polynomial

$$g_A = \prod_{m \in M} \Big(\sum_{i \in A} x_i - m\Big),$$

and the multilinear polynomial g_A^*. Then

$$g_A^*(X) = \prod_{m \in M} (|A \cap X| - m)$$

for any $X \subseteq V$, and $g_A^*(B) = 0$ for any distinct $A, B \in \mathcal{F}$. Note that $g_A^*(A) > 0$ for any $A \in \mathcal{F}$. We also put $h(x_1, x_2, \ldots, x_v) = \sum_{i=1}^{v} x_i - k$. Then $h(X) = |X| - k$ for any subset X of V, so $h(A) = 0$ for any $A \in \mathcal{F}$.

We claim that the set

$$\{g_A^* \colon A \in \mathcal{F}\} \cup \{(x_I h)^* \colon I \subseteq V, |I| \leq s-1\}$$

of multilinear polynomials is linearly independent. Since all these polynomials are in P_s, this would imply that

$$|\mathcal{F}| + \sum_{i=0}^{s-1} \binom{v}{i} \leq \dim P_s,$$

so $|\mathcal{F}| \leq \binom{v}{s}$.

Assume that

$$\sum_{A \in \mathcal{F}} \alpha_A g_A^* + \sum_{\substack{I \subseteq V \\ |I| \leq s-1}} \beta_I (x_I h)^* = 0,$$

for some rational coefficients α_A, β_I. Applying both sides of this equation to $B \in \mathcal{F}$, we obtain that $\alpha_B g_B^*(B) = 0$, so $\alpha_B = 0$. Therefore,

$$\sum_{\substack{I \subseteq V \\ |I| \leq s-1}} \beta_I (x_I h)^* = 0. \tag{1.3}$$

We will show by induction on $|I|$ that $\beta_I = 0$.

Note that for $J \subseteq V$, we have

$$x_I(J) = \begin{cases} 1 & \text{if } I \subseteq J, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Applying both sides of (1.3) to the empty set and using (1.4), we obtain $\beta_\varnothing = 0$. Let $1 \leq u \leq s-1$ and let $\beta_I = 0$ whenever $|I| \leq u-1$. Then we have

$$\sum_{\substack{I \subseteq V \\ u \leq |I| \leq s-1}} \beta_I (x_I h)^* = 0.$$

Applying both sides of this equality to a subset J of V of cardinality u and using (1.4), we obtain that $\beta_J = 0$. This completes the induction and the proof of the theorem. □

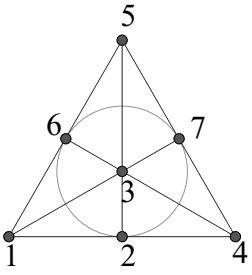

Figure 1.1 Fano Plane.

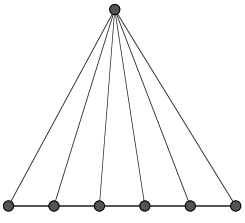

Figure 1.2 Pencil.

If $\mathcal{F}$ is the family of all s-subsets of the v-set V, then $|A \cap B| \in \{0, 1, \ldots, s-1\}$ for any distinct $A, B \in \mathcal{F}$ and $|\mathcal{F}| = \binom{v}{s}$, so the Ray-Chaudhuri–Wilson bound is sharp.

1.3. Symmetric designs and Ryser designs

By Fisher's Inequality (Theorem 1.1.1), the cardinality of a family of subsets of a v-set with one (nonzero) intersection number does not exceed v. In this section, we will consider families attaining this bound. The set of all $(v-1)$-subsets of a v-set is an example of such a family. We will give several less trivial examples.

Example 1.3.1. Let $V = \{1, 2, 3, 4, 5, 6, 7\}$ and let $\mathcal{F} = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$. Then $|\mathcal{F}| = |V|$ and $|A \cap B| = 1$ for any distinct $A, B \in \mathcal{F}$. This configuration is known as the *Fano Plane*. In Fig. 1.1, triples of points on lines or on the circle represent elements of the family $\mathcal{F}$. All these triples are regarded as *lines* in the Fano Plane.

Example 1.3.2. Let V be a finite set. Fix $x \in V$ and define $\mathcal{F}$ to be the family consisting of the set $V \setminus \{x\}$ and all 2-subsets of V containing x. Then $|\mathcal{F}| = |V|$ and $|A \cap B| = 1$ for any distinct $A, B \in \mathcal{F}$. Such a configuration is called a *pencil* (Fig. 1.2).

Example 1.3.3. Arrange the elements of a set V of cardinality 16 in a 4×4 array. For each $x \in V$, define a subset B_x of size 6 by taking the elements of V, other than x, which occur in the same row or column as x. It is easy to see that $|B_x \cap B_y| = 2$ for any distinct $x, y \in V$.

Let $V = \{1, 2, \ldots, \mathrm{v}\}$ be a set of cardinality v. Let λ be a positive integer and let $\mathcal{F}$ be a family of subsets of V such that $|A \cap B| = \lambda$ for any distinct $A, B \in \mathcal{F}$. For each $A \in \mathcal{F}$, denote by f_A the linear polynomial

$$f_A = \sum_{i \in A} x_i - \lambda. \tag{1.5}$$

In the proof of Theorem 1.1.1, we have shown that the set $\{f_A : A \in \mathcal{F}\} \cup \{1\}$ is linearly independent in the vector space P of linear polynomials in variables $x_1, x_2, \ldots, x_v$ (over the rationals).

Suppose now that the family $\mathcal{F}$ is of maximum size, i.e., $|\mathcal{F}| = v$. Then this set of polynomials is a basis of P. By expanding monomials x_i in this basis we will attempt to extract information which can be used to obtain a crude classification of the extremal case. For the next theorem we introduce the notion of the *replication number* that will be used throughout the book.

Definition 1.3.4. Let $\mathcal{F}$ be a family of subsets of a finite set V. For any $x \in V$, the number of elements of $\mathcal{F}$ which contain x is called the *replication number of x in $\mathcal{F}$*.

Theorem 1.3.5 (The Ryser–Woodall Theorem). *Let v and λ be positive integers and let $\mathcal{F}$ be a family of v subsets of a v-set V such that $|A \cap B| = \lambda$ for any distinct $A, B \in \mathcal{F}$. Then either all elements of V have the same replication number or they have exactly two distinct replication numbers r and r^* and $r + r^* = v + 1$. In the latter case, $2 \le r \le v - 1$ and $2 \le r^* \le v - 1$.*

Proof. Let $V = \{1, 2, \ldots, v\}$. If there is $A \in \mathcal{F}$ such that $|A| \le \lambda$, then $|A| = \lambda$ and $B \cap C = A$ for any distinct $B, C \in \mathcal{F}$. Therefore, each element of A has replication number $r = v$ and each element of $V \setminus A$ has replication number $r^* = 1$. Thus we have $r + r^* = v + 1$. From now on, we assume that $|A| > \lambda$ for each $A \in \mathcal{F}$. Then the set $\{f_A : A \in \mathcal{F}\} \cup \{1\}$ where the polynomials f_A are defined by (1.5), is a basis of the vector space P of linear polynomials in variables $x_1, x_2, \ldots, x_v$ over the rationals. We will expand the monomials x_i in this basis:

$$x_i = \sum_{A \in \mathcal{F}} \alpha_A^{(i)} f_A + \beta_i.$$

Applying both sides of this equation to $B \in \mathcal{F}$ and using (1.1), we obtain that $\alpha_B^{(i)} = (1 - \beta_i)/(|B| - \lambda)$ if $i \in B$ and $\alpha_B^{(i)} = -\beta_i/(|B| - \lambda)$ if $i \notin B$.

Therefore,

$$x_i = (1-\beta_i)\sum_{A\ni i}\frac{f_A}{|A|-\lambda} - \beta_i\sum_{A\not\ni i}\frac{f_A}{|A|-\lambda} + \beta_i. \tag{1.6}$$

Applying both side of (1.6) to the empty set and to the singleton $\{i\}$, we obtain:

$$0 = (1-\beta_i)(-\lambda)\sum_{A\ni i}\frac{1}{|A|-\lambda} - \beta_i(-\lambda)\sum_{A\not\ni i}\frac{1}{|A|-\lambda} + \beta_i, \tag{1.7}$$

$$1 = (1-\beta_i)(1-\lambda)\sum_{A\ni i}\frac{1}{|A|-\lambda} - \beta_i(-\lambda)\sum_{A\not\ni i}\frac{1}{|A|-\lambda} + \beta_i. \tag{1.8}$$

Subtract (1.7) from (1.8) to obtain that $\beta_i \neq 1$ and

$$\sum_{A\ni i}\frac{1}{|A|-\lambda} = \frac{1}{1-\beta_i}. \tag{1.9}$$

Equations (1.7) and (1.9) imply that $\beta_i \neq 0$ and

$$\sum_{A\not\ni i}\frac{1}{|A|-\lambda} = \frac{1}{\beta_i} - \frac{1}{\lambda}. \tag{1.10}$$

Adding (1.9) to (1.10) yields

$$\frac{1}{\lambda} + \sum_{A\in\mathcal{F}}\frac{1}{|A|-\lambda} = \frac{1}{\beta_i(1-\beta_i)}. \tag{1.11}$$

We can reduce (1.11) to a quadratic equation in β_i, whose coefficients do not depend on i. Therefore, β_i can have at most two distinct values, β and $\beta^* = 1-\beta$. If $\beta_i = \beta$, then applying both sides of (1.6) to the set V yields

$$1 = (1-\beta)r_i - \beta(v-r_i) + \beta,$$

where r_i is the replication number of i. This equation implies that $r_i = \beta(v-1)+1$. Similarly, if $\beta_i = \beta^*$, we obtain that $r_i = \beta^*(v-1)+1$. Thus, if all β_i are the same, then all points $i \in V$ have the same replication number. If β and β^* are the two distinct values of β_i, then the elements of V have two distinct replication numbers r and r^*. Since $\beta+\beta^* = 1$, we have $r + r^* = v+1$.

Since $r + r^* = v+1$, we have $r \geq 1$. If $r = 1$, then $r = \beta(v-1)+1$ implies $\beta = 0$ which is not the case. Therefore, if the family $\mathcal{F}$ has two replication numbers and $|A| > \lambda$ for all $A \in \mathcal{F}$, then the replication number of each element of V is greater than 1 and less than v. □

Let us now discuss the two possibilities that arise from the Ryser–Woodall Theorem.

Suppose first that $\mathcal{F}$ is a family of v subsets of a v-set V such that $|A \cap B| = \lambda$ for any distinct $A, B \in \mathcal{F}$ and all elements of V have the same replication number r. Fix $A \in \mathcal{F}$ and count in two ways pairs (x, B) with $B \in \mathcal{F}$, $B \neq A$, and $x \in A \cap B$. We obtain that $|A|(r-1) = \lambda(v-1)$. Therefore, if $\lambda > 0$, then all $A \in \mathcal{F}$ have the same cardinality. In this case, we will say that $(V, \mathcal{F})$ is a *symmetric* (v, k, λ)*-design*, where $k = |A|$ for all $A \in \mathcal{F}$. Counting in two ways pairs (x, A) with $A \in \mathcal{F}$ and $x \in A$ yields $k = r$. Examples 1.3.1 and 1.3.3 describe a symmetric $(7, 3, 1)$-design and a symmetric $(16, 6, 2)$-design, respectively. The precise definition and many other examples of symmetric designs will be given in the next chapter.

The second possibility arising from the Ryser–Woodall Theorem leads to the notion of a *Ryser design*.

Definition 1.3.6. Let v and λ be positive integers. A *Ryser design of index* λ *on* v *points* is a pair $(V, \mathcal{F})$ where V is a set of cardinality v and $\mathcal{F}$ is a family of v subsets of V (blocks) such that

(i) $|A \cap B| = \lambda$ for any distinct $A, B \in \mathcal{F}$;
(ii) $|A| > \lambda$ for all $A \in \mathcal{F}$;
(iii) there are blocks A and B such that $|A| \neq |B|$.

Example 1.3.2 describes a Ryser design of index 1 on v points. As will be shown in Section 14.1, pencils are the only possible Ryser designs of index 1 on v points.

1.4. Equidistant families of sets

We will now consider a distance function on the set of subsets of a finite set. It will measure how different two subsets are. The following definition introduces the famous *Hamming distance*.

Definition 1.4.1. Let V be a finite set. For any $X, Y \subseteq V$, define the Hamming distance $d(X, Y)$ to be the cardinality of the symmetric difference $X \triangle Y$ of X and Y.

The Hamming distance has the following properties that can be easily verified:

(i) $d(X, Y) \geq 0$; $d(X, Y) = 0$ if and only if $X = Y$;
(ii) $d(X, Y) = d(Y, X)$;
(iii) $d(X, Y) + d(Y, Z) \geq d(X, Z)$.

Definition 1.4.2. A family $\mathcal{F}$ of subsets of the set V is called *equidistant* if there exists a positive integer d such that $|A\triangle B| = d$ for any distinct A and B in $\mathcal{F}$.

In this section we will first find the maximum cardinality of an equidistant family of subsets of a v-set.

Theorem 1.4.3. *If $\mathcal{F}$ is an equidistant family of subsets of a finite set V of cardinality v, then $|\mathcal{F}| \le v + 1$.*

Proof. Let $\mathcal{F}$ be an equidistant family of subsets of the set $V = \{1, 2, \dots, v\}$, $|\mathcal{F}| \ge 2$, and let $d = |A\triangle B|$ for any distinct A and B in $\mathcal{F}$. With each $A \in \mathcal{F}$ we associate the following linear polynomial f_A in variables $x_1, x_2, \dots, x_v$:

$$f_A = \sum_{i\notin A} x_i - \sum_{i\in A} x_i + |A| - d. \tag{1.12}$$

Then, for any subset X of V (regarded as a (0, 1)-string),

$$f_A(X) = |A\triangle X| - d. \tag{1.13}$$

This implies that for any $A, B \in \mathcal{F}$,

$$f_A(B) = \begin{cases} 0 & \text{if} \quad B \ne A, \\ -d & \text{if} \quad B = A. \end{cases} \tag{1.14}$$

We claim that the set $\{f_A : A \in \mathcal{F}\}$ of linear polynomials is linearly independent (over the rationals). Indeed, if $\sum_{A\in\mathcal{F}} \alpha_A f_A = 0$ for some rational coefficients α_A, then, applying both sides of this equality to $B \in \mathcal{F}$ and using (1.14), we obtain that $\alpha_B(-d) = 0$, so $\alpha_B = 0$. Since the dimension of the vector space of linear polynomials in the variables $x_1, x_2, \dots, x_v$ equals $v + 1$, it follows that $|\mathcal{F}| \le v + 1$. □

Hadamard matrices provide examples of maximum cardinality equidistant families.

Definition 1.4.4. A *Hadamard matrix* is a square matrix with all entries equal to ± 1 and with any two distinct rows orthogonal.

For example,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is a Hadamard matrix of order 4.

Hadamard matrices arise in different areas of combinatorics. The order of a Hadamard matrix is 1 or 2 or a multiple of 4. One of the most famous open conjectures in combinatorics is that there exists a Hadamard matrix of every order that is divisible by 4. We will discuss Hadamard matrices at length in Chapter 4.

Example 1.4.5. Let $V = \{1, 2, \ldots, v\}$, and let $H = [h_{ij}]$ be a Hadamard matrix of order $v + 1$ with all entries in the last column equal to 1. For $i = 1, 2, \ldots, v + 1$, let $A_i = \{j \in V : h_{ij} = 1\}$. Then the family $\mathcal{F} = \{A_i : 1 \leq i \leq v + 1\}$ is equidistant. It is called a *Hadamard family*.

We will now show that this is the only possible example of a maximum size equidistant family.

Theorem 1.4.6. *Let $\mathcal{F}$ be an equidistant family of subsets of a v-set V. If $|\mathcal{F}| = v + 1$, then $\mathcal{F}$ is a Hadamard family.*

Proof. Let $|\mathcal{F}| = v + 1$, $|A \triangle B| = d$ for any distinct $A, B \in \mathcal{F}$, and let polynomials f_A be defined by (1.12). It was shown in the proof of Theorem 1.4.3 that the set $\{f_A : A \in \mathcal{F}\}$ of linear polynomials is linearly independent. Since $|\mathcal{F}| = v + 1$, this set is a basis of the vector space P of linear polynomials in $x_1, x_2, \ldots, x_v$. Expand the constant polynomial 1 in this basis:

$$1 = \sum_{A \in \mathcal{F}} \alpha_A f_A$$

for some rational coefficients α_A. Applying both sides of this equality to $B \in \mathcal{F}$, we derive that $\alpha_B(-d) = 1$, so $\alpha_B = -1/d$ for any $B \in \mathcal{F}$. Therefore, we have

$$\sum_{A \in \mathcal{F}} f_A = -d. \tag{1.15}$$

Applying both sides of (1.15) to the empty set and the set V, we obtain:

$$\sum_{A \in \mathcal{F}} (|A| - d) = -d$$

and

$$\sum_{A \in \mathcal{F}} (v - |A| - d) = -d.$$

Adding these equalities yields $(v + 1)(v - 2d) = -2d$, which implies $d = \frac{v+1}{2}$. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_{v+1}\}$. Define the following square matrix $H = [h_{ij}]$ of