1 The point particle

"We must proceed very slowly, for we are in a great hurry."
Statement at the beginning of the peace talks with the Palestinians

A number of the features of string theory are shared by the point particle. This is not too surprising as the point particle can be obtained in the limit as the string collapses to a point. Although one might think that the relativistic free point particle is a rather trivial system, it is a system with constraints and must be quantised with corresponding care. In this chapter we give the classical description of the point particle and then quantise it using first the Dirac method and then Becchi–Rouet–Stora–Tutin (BRST) quantisation techniques.

These steps are then repeated for the superparticle. There are two ways to incorporate supersymmetry into the point particle and these lead to different formulations that have, after quantisation, different physical states.

1.1 The bosonic point particle

1.1.1 The classical point particle and its Dirac quantisation

As the point particle moves through a Minkowski space-time of dimension D with coordinates $x^\mu,\ \mu = 0, 1, \ldots, D-1$, it sweeps out a one-dimensional curve called the world line which we choose to parameterise by τ. We may write the world line as $x^\mu(\tau)$. The motion of the point particle is taken to be so as to be an extremum of the action

$$A = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}, \tag{1.1.1}$$

where $\dot{x}^\mu \equiv dx^\mu/d\tau$ and $\eta_{\mu\nu}$ is the Minkowski metric, which in our conventions is given by $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, \ldots, +1)$. For a time-like particle, that is, one moving at less than the speed of light, $-ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - \sum_{i=1}^{D-1} dx^i dx^i > 0$, since in our units the speed of light is set to 1. For such a motion the quantity under the square root is positive. This is the reason for the minus sign under the square root. The proper time u of the particle is defined by $du^2 = -ds^2$ and we recognise that $A = -m \int du$. As a result, we conclude that the point particle moves so as to extremise its proper time.

The choice of parameterisation of the world line is of no physical significance and indeed the action of equation (1.1.1) is invariant under the reparameterisations $\tau \to \tau'(\tau)$. Such an infinitesimal transformation can be written as $\tau' = \tau - f(\tau)$, where f is a small quantity. The field x^μ is taken to be a scalar under reparameterisations, namely $x^{\mu\prime}(\tau') = x^\mu(\tau)$.

The infinitesimal variation of any field ϕ which is defined on a space-time labelled by ξ, all indices being suppressed, is defined to be $\delta\phi(\xi) = \phi'(\xi) - \phi(\xi)$. Therefore, an infinitesimal transformation acts on the field x^μ as

$$\delta x^\mu = x^{\mu\prime}(\tau) - x^\mu(\tau) = x^\mu(\tau + f) - x^\mu(\tau) = f(\tau)\dot{x}^\mu. \tag{1.1.2}$$

We may write the above action in an alternative, but classically equivalent way, namely

$$A = \frac{1}{2}\int d\tau\{e^{-1}\dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu} - m^2 e\}, \tag{1.1.3}$$

where x^μ and e are independent fields. This action is also reparameterisation invariant under the transformation $\tau' = \tau - f(\tau)$, which acts on the above fields as the infinitesimal transformations

$$\delta x^\mu = f\dot{x}^\mu,\ \delta e = f\dot{e} + \dot{f}e. \tag{1.1.4}$$

We recognise the first term in equation (1.1.3) as D scalar fields coupled to one-dimensional gravity; e is the einbein on the one-dimensional world line and the metric is given by $g_{\tau\tau} = -e^2$.

It is often useful to introduce explicitly the momentum p^μ and write the action of equation (1.1.3) in a first order form, namely

$$A = \int d\tau\left\{\dot{x}^\mu p^\nu\eta_{\mu\nu} - \frac{e}{2}(p^\mu p^\nu\eta_{\mu\nu} + m^2)\right\}. \tag{1.1.5}$$

Eliminating p^μ by its algebraic equation of motion in this action we recover the action of equation (1.1.3), demonstrating that these two actions are equivalent. The equations of motion of the action of equation (1.1.3) are

$$\frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0,\ e^2m^2 + \dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu} = 0. \tag{1.1.6}$$

Substituting e from its equation of motion into the action of equation (1.1.3), we recover the action of equation (1.1.1), while the first equation becomes the equation of motion of the action of equation (1.1.1).

All the above actions are Poincaré invariant, but the second two actions have the advantage that they can be used even in the massless case, that is, $m = 0$. In this case, the actions of equation (1.1.3) and (1.1.5) are also invariant under space-time conformal transformations. Indeed, it is straightforward to verify that the action of equation (1.1.3), with $m = 0$, is invariant under $\delta x^\mu = w^\mu,\ \delta e = (2/D)e\partial_\nu w^\nu$ provided $\partial_\mu w_\nu + \partial_\nu w_\mu = (2/D)\eta_{\mu\nu}\partial_\rho w^\rho$. We recognise the latter condition as that required for a reparameterisation to preserve the Minkowski space line element up to an arbitrary scale factor, in other words a conformal transformation. We refer the reader to chapter 8 for further details of conformal transformations.

To analyse the point particle, we may start from any of the above actions. Let us first take the action of equation (1.1.1) Taking τ as our time evolution parameter, the corresponding canonical momentum is given by

$$p^\mu = \frac{\partial L}{\partial\dot{x}_\mu(\tau)} = \frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu}}}, \tag{1.1.7}$$

where L is the Lagrangian and is given by $L = -m\sqrt{-\dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu}}$.

We find by inspection that the momenta automatically satisfy the constraint

$$\phi \equiv p^\mu p_\mu + m^2 = 0 \tag{1.1.8}$$

and so there are fewer momenta than coordinates. This can be viewed as a consequence of the reparameterisation invariance of the action. The Hamiltonian

$$H = p^\mu \dot{x}_\mu - L \tag{1.1.9}$$

vanishes once we substitute for p^μ. As the Hamiltonian is the generator of time translations and so the motion of the system, we might appear, at first sight, to have no dynamics.

The method of dealing with such a constrained system was given by Dirac in [1.1] and we encourage the reader to consult this reference. However, we give a summary of the Dirac method in appendix A that will be sufficient to completely understand the following sections. The reader who is unfamiliar with this method may wish to read appendix A before going further. However, in the discussion below the steps are rather natural and for those in a hurry it can be followed without an additional reading.

We now apply the Dirac method of quantisation to the point particle. The Poisson brackets that involve x^μ and p^μ are given by

$$\{x^\mu, x^\nu\} = 0 = \{p^\mu, p^\nu\}, \quad \{x^\mu, p^\nu\} = \eta^{\mu\nu}. \tag{1.1.10}$$

We take the Hamiltonian, which by usual methods vanishes, to be proportional to the constraint ϕ multiplied by $v(\tau)$, which is an arbitrary function of τ. It is given by

$$H = v(\tau)(p^\mu p_\mu + m^2). \tag{1.1.11}$$

Continuing the Dirac procedure we must demand that the constraint is preserved in time. However, in this case, we find that $(d/d\tau)\phi = \{\phi,\ H\} = 0$ and so there is no new constraint. Hence we have a system which has only one constraint which obviously obeys $\{\phi\ , \phi\} = 0$, and so it is a first class constraint in the language of Dirac.

The Hamiltonian H generates time translations and so the equations of motion are

$$\frac{dx^\mu}{d\tau} = \{x^\mu, H\} = 2v(\tau)p^\mu, \ \frac{dp^\mu}{d\tau} = \{p^\mu, H\} = 0. \tag{1.1.12}$$

We note that reparameterisations change the dependence of the coordinates x^μ on time while time evolution shifts the time dependence. As such, reparameterisations and time evolution have much in common and in particular $\tau \to \tau +$ constant is a reparameterisation which can also be viewed as a time evolution. This is why the right-hand side of the above equations of motion resembles a reparameterisation when multiplied by an appropriate parameter.

To quantise the theory we make the usual transition, according to the Dirac rule, from Poisson brackets to commutators, with an appropriate factor of $i\hbar$. The commutators for the coordinates and momenta are then given by $[x^\mu, x^\nu] = 0 = [p^\mu, p^\nu]$ and $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}$. These commutators are represented by the replacements

$$x^\mu \to x^\mu; \quad p^\mu \to -i\hbar\frac{\partial}{\partial x_\mu}. \tag{1.1.13}$$

Setting $\hbar = 1$, the constraint becomes

$$\hat{\phi} = (-\partial^2 + m^2). \tag{1.1.14}$$

This is no longer an algebraic condition, but a differential operator. To proceed further, we consider the particle to be described by a wavefunction, or field, $\psi(x^\mu, \tau)$ and we impose the constraint

$$\hat{\phi}\psi = (-\partial^2 + m^2)\psi = 0. \tag{1.1.15}$$

We also impose the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial\tau} = H \cdot \psi. \tag{1.1.16}$$

However, the right-hand side of this equation vanishes once the constraint ϕ is imposed and we find that ψ is independent of τ. We recognise the usual formulation of a second-quantised spin-0 particle namely, the τ dependence has disappeared and we are left with the Klein–Gordon equation.

Let us now briefly consider starting from the alternative action of equation (1.1.3). The momentum conjugate to x^μ is $p^\mu = e^{-1}\dot{x}^\mu$, but the momentum p_e conjugate to e vanishes. We take this latter condition as a constraint: $r \equiv p_e = 0$. The Hamiltonian is found to be $H = (e/2)(p^\mu p_\mu + m^2) + s(\tau)p_e$, where s is an arbitrary function of τ. The non-vanishing fundamental Poisson brackets are $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$, $\{e, p_e\} = 1$. Insisting that the time development of the constraint $r = 0$ should vanish implies that $\dot{r} = \{r, H\} = \frac{1}{2}(p^\mu p_\mu + m^2) = 0$. Hence, we recover the constraint of equation (1.1.7) and the Hamiltonian vanishes as it is proportional to the constraints. It is easy to verify that there are no further constraints.

To quantise the system we proceed much as before. We turn the Poisson brackets into commutators with an $i\hbar$ factor and adopt a Schrödinger representation. The constraints are then imposed on the wavefunction, which depends on the coordinates x^μ and e. The constraint $p^\mu p_\mu + m^2 = 0$ becomes the Klein–Gordon equation, while the other constraint states that the wavefunction does not depend on e. The Schrödinger equation simply states that the wavefunction does not depend on τ. Hence, we arrive at the same quantum system.

Although we may not implement a gauge choice naively in the action, we may use it on the equations of motion of any system. We note that the equations of motion of the action of equation (1.1.3) given in equation (1.1.6) can be simplified by a suitable gauge choice. Indeed, we may use the reparameterisation invariance of equation (1.1.4) to choose $e = 1$, whereupon they become

$$\ddot{x}^\mu = 0, \ \dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu} + m^2 = 0. \tag{1.1.17}$$

We note that the second equation, when expressed in terms of momenta, is just the constraint found above. The constraint $\dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu} + m^2 = 0$ is none other than the condition that the energy-momentum tensor of the one-dimensional system in the absence of gravity should vanish. We may read off the energy-momentum tensor by substituting $e = 1 + h$ into the action of equation (1.1.3) expanding in terms of h and taking the coefficient of the term linear in h. Another possible gauge choice is to take $\dot{x}^0 = 1$. This is called the static gauge and an analogue of it will be used extensively when we come to discuss branes.

1.1.2 The BRST quantization of the point particle

We now wish to apply the BRST approach to the point particle. The BRST transformations were found in [1.2], they were further developed in [1.3] and the relation between the BRST charge and the physical states were found in [1.4]. This approach will be particularly important for the string. The reader may wish to first read appendix A, where the BRST formulation of Yang–Mills theory is given and the general procedure is explained. We begin with the action of equation (1.1.3), which is world-line reparameterisation invariant under the transformations of equation (1.1.2). Corresponding to the one local invariance with parameter $f(\tau)$, we introduce the ghost $c(\tau)$ and anti-ghost $b(\tau)$, which are both Grassmann odd.

The BRST transformations of the original fields are found by the substitution $f(\tau) \to \Lambda c(\tau)$, where Λ is a Grassmann odd BRST parameter, into the original local transformations of equation (1.1.4). The result is

$$\delta x^\mu = (\Lambda c)\dot{x}^\mu, \quad \delta e = \frac{d}{d\tau}((\Lambda c)e). \tag{1.1.18}$$

We choose c to be Hermitian, but Λ is taken to be anti-Hermitian in order that Λc be real. The standard rule for taking the complex conjugate of two Grassmann odd variables is to reverse their order and then take their individual complex conjugates; thus

$$(\Lambda c)^* = c^*\Lambda^* = -\Lambda^* c^* = \Lambda c. \tag{1.1.19}$$

The transformation law of the ghost is given by

$$\delta c = (\Lambda c)\dot{c}. \tag{1.1.20}$$

Under an infinitesimal reparameterisation x^μ transforms as $\delta x^\mu = f\dot{x}^\mu$; carrying out the commutation of two infinitesimal reparameterisations f_1 and f_2 yields a third reparameterisation with parameter

$$f_{12} = (-f_1\dot{f}_2 + f_2\dot{f}_1). \tag{1.1.21}$$

Following the prescription in appendix A, we find that $\tilde{f}_{12} = 2\Lambda c\dot{c}$ and so equation (1.1.20).

To the fields x^μ, e, c, b we add the Lagrange multiplier λ (called B in appendix A). The anti-ghost b transforms into the multiplier λ, namely,

$$\delta\lambda = 0, \quad \delta b = \Lambda\lambda. \tag{1.1.22}$$

Although the above may seem like a cookery book recipe, we have arrived at one of the desired results, namely a set of nilpotent transformations. For example, two transformations on c are given by

$$\begin{aligned}\delta_{\Lambda_1}\delta_{\Lambda_2} c &= \delta_{\Lambda_1}\{(\Lambda_2 c)\dot{c}\} \\ &= \Lambda_2((\Lambda_1 c)\dot{c})\,\dot{c} + \Lambda_2 c\frac{d}{d\tau}\{(\Lambda_1 c)\dot{c}\} = 0.\end{aligned} \tag{1.1.23}$$

We now choose the gauge fixing function to be

$$G = \ln e. \tag{1.1.24}$$

Setting $G = 0$ sets $e = 1$. Consequently, we should add to the original action of equation (1.1.3) the gauge fixing term

$$A^{gf} = \int d\tau \lambda \ \ln e. \tag{1.1.25}$$

A BRST invariant action is given by

$$\begin{aligned} A^{BRST} &= A^{orig} + A^{gf} + A^{gh} \\ &= \tfrac{1}{2} \int d\tau \{e^{-1} \dot{x}^{\mu} \dot{x}^{\nu} \eta_{\mu\nu} - m^2 e\} + \int d\tau \lambda \ \ln e - \int d\tau b D_{\tau} c, \end{aligned} \tag{1.1.26}$$

where A^{orig} is the action of equation (1.1.3) and A^{gf} is that of equation (1.1.25). Due to its original reparameterisation invariance A^{orig} is automatically BRST invariant. We find by cancelling the variations of A^{gf} that

$$A^{gh} = -\int d\tau b D_{\tau} c, \tag{1.1.27}$$

where $D_{\tau} c = \dot{c} + (d \ln e / d\tau) c$. An alternative way of arriving at the above result is to note that under a BRST transformation

$$\delta_{\Lambda} \{b(\ln e)\} = \Lambda(\lambda(\ln e) - b D_{\tau} c), \tag{1.1.28}$$

and use the nilpotency of δ_{Λ} to establish the BRST invariance of $A^{gf} + A^{gh}$.

The quantum theory is then given by the functional integral

$$\int De Dx^{\mu} Dc Db D\lambda \exp(i A^{BRST}).$$

In this functional integral, we can carry out the λ integration which sets $e = 1$, whereupon the BRST action becomes

$$A^{final} = \int d\tau \left(\tfrac{1}{2} \dot{x}^{\mu} \dot{x}^{\nu} \eta_{\mu\nu} - \tfrac{1}{2} m^2 - b\dot{c} \right). \tag{1.1.29}$$

This result is still BRST invariant; however, for δb we must substitute the value of λ given by the e equation of motion with $e = 1$, that is,

$$\delta b = \Lambda \left(\frac{\dot{x}^{\mu} \dot{x}_{\mu}}{2} + \frac{m^2}{2} - \frac{d}{d\tau}(bc) \right), \tag{1.1.30}$$

the other variations being unchanged.

As the action of equation (1.1.29) is BRST invariant, we can in the standard way deduce the associated Noether current Q which, in this one-dimensional case, is also the BRST charge. We find that it is given by

$$Q = c(p^{\mu} p_{\mu} + m^2), \tag{1.1.31}$$

where $p^{\mu} = \dot{x}^{\mu}$ is the momentum conjugate to x^{μ}. We take the definition of momenta for Grassmann odd variables to be left differentiation of the action by the coordinate. Hence, the momentum for the coordinate c is given by

$$\frac{\vec{\partial}}{\partial \dot{c}} A = b. \tag{1.1.32}$$

We could also take b as our coordinate and then c would be the corresponding momentum.

We now give a Poisson bracket suitable for a general system which contains Grassmann even and odd variables. Given a system with coordinates q_A and corresponding momenta p_A, some of which may be Grassmann odd, we define the Poisson bracket of two functions f and g of q_A and p_A as

$$\{f, g\}_{PB} = \sum_A \left\{ f \frac{\overleftarrow{\partial}}{\partial q_A} \frac{\overrightarrow{\partial}}{\partial p_A} g - (-1)^{fg} g \frac{\overleftarrow{\partial}}{\partial q_A} \frac{\overrightarrow{\partial}}{\partial p_A} f \right\}, \tag{1.1.33}$$

where

$$(-1)^{fg} = \begin{cases} -1 & \text{if } f \text{ and } g \text{ are Grassmann odd,} \\ 1 & \text{otherwise.} \end{cases}$$

It satisfies the relations

$$\begin{aligned} &\{f, g\}_{PB} = -(-1)^{fg}\{g, f\}_{PB}, \ \{f, gk\}_{PB} = (-1)^{fg} g\{f, k\}_{PB} + \{f, g\}_{PB} k, \\ &\{f_1 + f_2, g\}_{PB} = \{f_1, g\}_{PB} + \{f_2, g\}_{PB} \end{aligned} \tag{1.1.34}$$

as well as a generalised Jacobi identity.

Returning to the point particle and the action of equation (1.1.29). We find that the resulting non-zero Poisson brackets for the coordinates and momenta are

$$\{x^\mu, p^\nu\}_{PB} = \eta^{\mu\nu}, \ \{c, b\}_{PB} = 1. \tag{1.1.35}$$

The Hamiltonian associated to the action of equation (1.1.29) is given by

$$H = p^\mu p_\mu + m^2. \tag{1.1.36}$$

The BRST charge is the generator of transformations in the usual sense that

$$\delta \bullet = \{\bullet, \Lambda Q\}_{PB}, \tag{1.1.37}$$

where $\bullet$ is any field. The reader may verify that, on-shell, these transformations agree with those previously given and are an invariance of the Hamiltonian equations of motion. We note that the Q satisfies the equation $\{Q, Q\}_{PB} = 0$.

To quantise the system we apply the Dirac rule

$$\{,\}_{PB} \to \begin{cases} \dfrac{1}{i\hbar} & \{,\} \text{ for two odd quantities,} \\ \dfrac{1}{i\hbar} & [,] \text{ otherwise} \end{cases} \tag{1.1.38}$$

to the Poisson brackets. In the above, $\{A, B\} \equiv AB + BA$ and $[A, B] \equiv AB - BA$. The use of the symbol $\{,\}$ in the quantum theory should not be confused with the classical Poisson bracket used in the other sections in this book where it is not generally given the subscript PB. Consequently, we must demand

$$[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}, \quad \{c, b\} = i\hbar. \tag{1.1.39}$$

We may use the generalisation of the Schrödinger representation:

$$x^\mu \to x^\mu, \quad c \to c \quad p^\mu \to -i\hbar\frac{\partial}{\partial x^\mu} \quad b \to i\hbar\frac{\partial}{\partial c}. \tag{1.1.40}$$

In checking the appearance of is, it is important to remember that b is anti-Hermitian.

The BRST charge now becomes the operator

$$Q = c(-\partial^2 + m^2) \tag{1.1.41}$$

and it is obviously nilpotent, that is, $Q^2 = 0$ as $c^2 = 0$.

We now consider wavefunctions of the coordinates, that is, $\Psi(x^\mu, c)$. Taylor expanding in c, the wavefunction has the form

$$\Psi(x^\mu, c) = \psi(x^\mu) + c\phi(x^\mu) \tag{1.1.42}$$

as c is Grassmann odd. In the BRST formalism, physical states are taken to satisfy the condition

$$Q\Psi = 0. \tag{1.1.43}$$

This condition is equivalent to

$$(-\partial^2 + m^2)\psi = 0, \tag{1.1.44}$$

which is the usual Klein–Gordon result. Two physical states are equivalent if they differ by a term of the form $Q\Omega$. We may take $\Omega(x^\mu, c)$ to be of the form $\Omega(x^\mu, c) = a(x^\mu) + cb(x^\mu)$ and so $Q\Omega(x^\mu, c) = c(-\partial^2 + m^2)a(x)$. Thus two fields ϕ are equivalent if they differ by a term of the form $(-\partial^2 + m^2)a(x)$ for any a and so any ϕ is equivalent to the trivial field 0. Thus the physical states are described by only the field ψ subject to equation (1.1.44), in other words the well-known result.

We observe that the results of the usual quantisation carried out in section (1.1.1) can be rewritten in terms of the BRST formalism. For example, equation (1.1.15) can be written as $Q\Psi = 0$ if the field Ψ is subject to $b\Psi = 0$. The latter condition sets ϕ to zero. We also note that the equation $Q\Psi = 0$ has a local invariance under $\delta\Psi = Q\Lambda$, where now Λ is a arbitrary function of space-time. At first sight, these observations could be regarded as a cumbersome way of describing the point particle. However, with hindsight, and after the discovery of an analogous equation for the string, it was realised that what had looked like an artificial manipulation of the BRST formalism had a deeper significance. The meaning of the above statements will become clearer when we study gauge covariant string theory in chapter 12.

The method of BRST quantization originated [1.2] in the context of Yang–Mills theory, which is still the prototype example of how to proceed. The systematic use of the BRST charge, for general systems with first class constraints, was carried out in [1.3]. For some reviews of this procedure, see [1.4]. It must be stated, however, that the BRST method, as with any quantisation method, is more like an art than a science. Its justification is that the final result, namely a nilpotent set of transformations and an invariant action, usually defines a quantum theory which is unitary and whose physical observables are independent of how the gauge was fixed.

1.2 The super point particle

The key to finding systems whose quantisation leads to particles with spin is the introduction of Grassmann odd degrees of freedom [1.5, 1.6]. There are, however, two different ways to proceed: we may extend the ordinary point particle by encoding either a world-line supersymmetry or a space-time supersymmetry. These two formulations, called the spinning particle [1.7, 1.8] and the Brink–Schwarz particle [1.9], respectively, look very different and indeed lead to different physics. While the former may be quantised straightforwardly, the covariant quantisation of the latter presents formidable problems. In what follows we explain both formulations beginning with the spinning particle. We also explain how twistors can be used to give alternative formulations of the super point particle that avoid the quantisation problems found in one of the formulations. It is interesting to compare the discovery of the supersymmetric formulation of the spinning particle with the corresponding development, as discussed in chapter 6, of the superstring with world-sheet supersymmetry whose supersymmetric action was found very shortly afterwards.

1.2.1 The spinning particle

The massless bosonic particle is described by D fields x^μ, $\mu = 0, 1, \ldots, D-1$ coupled to one-dimensional gravity with einbein e. To extend this to a formulation with world-line supersymmetry we begin with a theory of D fields x^μ and their superpartners χ^μ which is invariant under rigid supersymmetry. We will then couple this system to one-dimensional, or world-line, supergravity which is described by the graviton and its superpartner, the gravitino ψ.

Let us therefore consider the action

$$A^F = \tfrac{1}{2}\int d\tau\,(\dot{x}^\mu \dot{x}^\nu - i\chi^\mu \dot{\chi}^\nu)\eta_{\mu\nu}. \tag{1.2.1}$$

We take x^μ and χ^μ to be Grassmann even and odd, respectively. We recall that in the classical theory, Grassmann even objects commute with Grassmann even and Grassmann odd objects, but that Grassmann odd objects anti-commute with Grassmann odd objects, that is, $\chi^\mu\chi^\nu = -\chi^\nu\chi^\mu$. The action of equation (1.2.1) is invariant under rigid time translations

$$\delta x^\mu = a\dot{x}^\mu,\ \delta\chi^\mu = a\dot{\chi}^\mu \tag{1.2.2}$$

and rigid supersymmetry, whose transformations are given by

$$\delta x^\mu = i\epsilon\chi^\mu, \quad \delta\chi^\mu = \dot{x}^\mu\epsilon, \tag{1.2.3}$$

where ϵ is Grassmann odd. We choose ϵ and χ^μ to be real, that is, $\epsilon^* = \epsilon$, $\chi^{\mu *} = \chi^\mu$. The commutator of two such transformations is found to be

$$[\delta_1, \delta_2]\bullet = 2i\epsilon_2\epsilon_1 \frac{d}{d\tau}\bullet \tag{1.2.4}$$

acting on either x^μ or χ^μ, which are denoted by $\bullet$ in the above equation. We recognise the result as a time translation of magnitude $2i\epsilon_2\epsilon_1$.

The supersymmetry current j is given by

$$j = \dot{x}^\mu\chi^\nu\eta_{\mu\nu}. \tag{1.2.5}$$

One standard method of finding the current corresponding to any rigid symmetry is to let the parameter of the symmetry become local, that is, space-time dependent, and then compute the variation of the action. Since the action is invariant when the parameter is constant, the variation of the action must contain the space-time derivative of the parameter times a quantity that is just the current. This identification follows from the fact that any variation of the action is given by the equation of motion multiplied by the field variation and so it must vanish when the equations of motion are enforced. As a result, in the case of the above variation we conclude that the object identified as the current is indeed conserved if the equations of motion hold. In the case under consideration here we let $\epsilon \to \epsilon(\tau)$ and write the variation of the action as $\delta A = i \int d\tau ((d/d\tau)\epsilon) j$. Carrying out this calculation we find the supersymmetry current of equation (1.2.5).

The coupling of the action of equation (1.2.1) to world-line supergravity (e, ψ) is given by

$$A = \frac{1}{2} \int d\tau \left(e^{-1}\dot{x}^\mu \dot{x}^\nu - i\chi^\mu \dot{\chi}^\nu - \kappa e^{-1} i\psi \chi^\mu \dot{x}^\nu\right)\eta_{\mu\nu} \tag{1.2.6}$$

and the local supersymmetry transformations which leave it invariant are

$$\begin{aligned} \delta x^\mu &= i\epsilon\chi^\mu, \quad \delta\chi^\mu = \left(\dot{x}^\mu - \frac{\kappa}{2} i\psi\chi^\mu\right)\epsilon e^{-1}, \\ \delta e &= i\kappa\epsilon\psi, \quad \delta\psi = \frac{2}{\kappa}\dot{\epsilon}, \end{aligned} \tag{1.2.7}$$

where the supersymmetry parameter ϵ is an arbitrary function of τ. It is also invariant under reparameterisation symmetry $\delta x^\mu = k\dot{x}^\mu$, $\delta\chi^\mu = k\dot{\chi}^\mu$, $\delta e = d(ke)/d\tau$ and $\delta\psi = d(k\psi)/d\tau$, where k is an arbitrary function of τ.

We now explain how this result is found using the Noether method since it provides a simple example which illustrates most of the points required to construct supergravity theories using this method. An explanation of the Noether technique is given in chapter 13. The Noether technique is not required again for the super point particle and the reader who is not interested in this derivation may skip the following discussion and resume at equation (1.2.24).

We start with the action of equation (1.2.1) with the rigid, that is, constant, supersymmetry transformations of equation (1.2.3) and the linearized supergravity fields h and ψ which have the rigid supersymmetry transformations

$$\delta h = i\epsilon\psi, \quad \delta\psi = 0. \tag{1.2.8}$$

The supergravity fields have the Abelian local transformations

$$\delta h = \dot{g}, \quad \delta\psi = \dot{\eta}, \tag{1.2.9}$$

where g and η are arbitrary functions of τ, which are Grassmann even and odd, respectively. Despite the unusual appearance of the transformations of equations (1.2.8) it is trivial to verify that they close provided one allows for the occurrence of the transformations of equations (1.2.9). Of course, h and ψ can be gauged away using these latter transformations corresponding to the fact there is no gravity, or supergravity, in one dimension. In what follows we will suppress the μ, ν indices until we have found the local result.

We now let the previously constant supersymmetry parameter ϵ become τ-dependent. The action A^F is no longer invariant, but

$$A^1 = A^F - i \int d\tau \frac{\kappa}{2} \psi j \tag{1.2.10}$$