

## RECIPROCITY IN ELASTODYNAMICS

The reciprocity theorem has been used for over 100 years to establish interesting and useful relations between different loading states of a body. In a well-known application, reciprocity relations have been used to formulate problems in a way suitable for numerical calculation by the boundary element method.

This book presents a discussion of current and novel uses of reciprocity relations for the determination of elastodynamic fields. The author, who is internationally distinguished for his contributions to theoretical and applied mechanics, presents a novel method to solve for wave fields by the reciprocity of the actual field with a so-called virtual wave, shedding new light on the use of reciprocity relations for dynamic fields in an elastic body. Professor Achenbach describes the use of reciprocity in acoustics, in one-dimensional elastodynamics, and in two- and three-dimensional elastodynamics. Integral representations and integral equations for application of the boundary element method are discussed in some detail, with applications to scattering by inhomogeneities and related inverse problems. Chapters are also devoted to reciprocity for coupled acousto-elastic systems, such as elastic bodies submerged in a fluid, and to reciprocity for piezoelectric systems.

The material presented in the book is relevant to several fields in engineering and applied physics. Examples are ultrasonics for medical imaging and non-destructive evaluation, acoustic microscopy, seismology, exploratory geophysics, structural acoustics, the response of structures to high-rate loads and the determination of material properties by ultrasonic techniques.

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# RECIPROCITY IN ELASTODYNAMICS

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## Preface

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Reciprocity relations are among the most interesting and intriguing relations in classical physics. At first acquaintance these relations promise to be a goldmine of useful information. It takes some ingenuity, however, to unearth the nuggets that are not immediately obvious from the formulation. In the theory of elasticity of solid materials the relevant reciprocity theorem emanated from the work of Maxwell, Helmholtz, Lamb, Betti and Rayleigh, towards the end of the nineteenth century, and several applications have appeared in the technical literature since that time. This writer has always believed, however, that more information than is generally assumed can be wrested from reciprocity considerations. I have wondered in particular whether reciprocity considerations could be used to actually determine by analytical means the elastodynamic fields for the high-rate loading of structural configurations. I have explored this question for a number of problems and obtained the actual fields generated by loading from a reciprocity relation in conjunction with an auxiliary solution, a free wave called the “virtual” wave. These recent results comprise an important part of the book.

To my knowledge, the topic of reciprocity in elastodynamics has not been discussed in a comprehensive manner in the technical literature. It is hoped that this book will fill that void. Various forms of the reciprocity theorem are presented, with an emphasis on those for time-harmonic fields, together with numerous applications, general and specific, old and new.

The book should be of interest to research workers in such fields as the ultrasonics of solids, particularly the detection of defects and the determination of elastic constants, seismology, exploratory geophysics, the dynamic response of structures and structural acoustics.

Parts of the book were read by graduate students in a course on wave propagation in elastic solids. My colleague John Harris also read a number of chapters,

as did David Feit and A. T. de Hoop. Their comments are gratefully acknowledged.

A special word of thanks goes to Linda Kearfott who typed and retyped the manuscript as the material was arranged, rearranged and revised.

Over the years my work in the area of this book and in related areas has been consistently supported by the Office of Naval Research (ONR). The monitor of my research project in recent years has been Dr Y. N. Rajapakse. The ONR funding is gratefully acknowledged.

The constant support of my wife, Marcia, made it possible to complete the book. I can never thank her enough for all she has given me.