1
Introduction

1.1 Reciprocity

Reciprocity is a good thing. Something is given and something else, equally or more valuable, is returned. So it is in reciprocity for states of deformation of elastic bodies. What is received in return is the main benefit from the reciprocal relationship. From a known solution to one loading case, some important aspect of, or the complete solution to, another loading case is returned. The return is, however, not always a complete solution, but sometimes an equation for computing such a solution.

For dynamic systems the concept of reciprocity goes back to the nineteenth century. A pertinent reciprocity theorem was first formulated by von Helmholtz (1860). Lord Rayleigh (1873, 1877), subsequently derived a quite general reciprocity relation for the time-harmonic motion of a linear dynamic system with a finite or infinite number of degrees of freedom. Rayleigh's formulation included the effects of dissipation. In a later work Lamb (1888) attributed the following general reciprocity theorem to von Helmholtz (1886):

> Consider any natural motion of a conservative system between two configurations A and A' through which it passes at times t and t' respectively, and let $t' - t = \tau$. Let $q_1, q_2 \ldots$, be the coordinates of the system, and $p_1, p_2, \ldots$ the component momenta, at time t, and let the values of the same quantities at time t' be distinguished by accents. As the system is passing through the configuration A, let a small impulse δp_r of any type be given to it; and let the consequent alteration in any coordinate q_s after the time τ be denoted by $\delta q_s'$. Next consider the *reversed* motion of the system, in virtue of which it would, if undisturbed, pass from the configuration A' to the configuration A in the time τ. Let a small impulse $\delta p_s'$ be applied as it is passing through the configuration A', and let the consequent change in the coordinate q_r, after a time τ, be δq_r. The reciprocity theorem asserts that
>
> $$\delta q_r : \delta p_s' = \delta q_s' : \delta p_r. \tag{1.1.1}$$

> If the coordinates q_r, q_s be of the same kind (e.g., both lines or both angles), the statement of the theorem may be simplified by supposing $\delta p'_s = \delta p_r$, in which case
>
> $$\delta q_r = \delta q'_s. \tag{1.1.2}$$
>
> In words, the change produced in the time τ by a small initial impulse of any type in the coordinate of any other (or of the same) type, in the *direct* motion, is equal to the change produced in the same time by a small initial impulse of the second type in the coordinate of the first type, in the *reversed* motion.

Lamb (1888) asserted that the reciprocity theorems of von Helmholtz (1886), formulated in his paper on the theory of least action and in earlier papers on acoustics and optics, and of Lord Rayleigh in acoustics, were particular cases of a general result derived in his 1888 paper.

Reciprocity theorems in elasticity theory provide a relation between displacements, traction components and body forces for two different loading states of a single body or two bodies of the same geometry. As discussed by Love (1892), for the elastostatic case the principal theorem is due to Betti (1872). A more general theorem, which includes the elastodynamic case, was given by Raleigh (1873). Statements of elastodynamic reciprocity theorems using contemporary notation can be found in, among others, the books by Achenbach (1973), Achenbach, Gautesen and McMaken (1982) and de Hoop (1995).

In the present text we concern ourselves with applications of reciprocity to time-harmonic elastodynamic states. The reciprocity theorem can be used to obtain a number of interesting relations between two such states. A well-known result is the relation between the magnitudes and directions of the forces and displacements at the points of application of two forces applied to an elastic body of finite dimensions or infinite extent. An important application of the reciprocity theorem produces boundary integral equations. If, for bodies with boundaries subjected to loads or displacements, the solution to a concentrated load in an unbounded solid of the same material is known then an equation (a boundary integral equation) for fields on the boundary, complementary to the applied fields, can be obtained, as shown in the sequel.

Of course, it would be desirable to obtain more than just the relations between elastic states or boundary integral equations from reciprocity relations. The purpose of the present work is therefore to give also direct applications to the computation of elastodynamic displacement fields. As will be discussed, it is possible to obtain a complete solution for certain configurations and concentrated loading cases by the use of elastodynamic reciprocity. The selected auxiliary solution for these cases we call a "virtual" wave. This is a wave motion that satisfies appropriate conditions on the boundaries and is a solution of the elastodynamic equations. It is shown that combining the desired solution as

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state A with a virtual wave as state B provides explicit expressions for state A. By using a virtual wave for other cases, particularly problems involving scattering by obstacles in waveguides, simple general expressions can be obtained for reflection and transmission coefficients and for scattering coefficients.

A formal definition of an elastodynamic reciprocity theorem may be given as follows: "A reciprocity theorem relates, in a specific manner, two admissible elastodynamic states that can occur in the same time-invariant linearly elastic body. Each of the two states can be associated with its own set of time-invariant material parameters and its own set of loading conditions. The domain to which the reciprocity theorem applies may be bounded or unbounded."

Elastodynamics covers both vibrations, which are standing waves, and propagating waves. The applications of reciprocity considerations discussed in this book are, however, primarily to propagating waves.

The propagation of mechanical waves is a common occurrence in nature. The simplest example is sound in an acoustic medium, which is a special case of elastodynamics. Not all sound is audible to the human ear. Sound with frequencies too low (infrasound), or too high (ultrasound, above 20 000 Hz), cannot normally be heard. Pierce (1981) briefly discussed the history of acoustics. Observation of the propagation of water waves, produced for example by a pebble dropped in a pond, led the Greeks and Romans to speculate that sound was a wave phenomenon. Aristotle (384–22 BC) stated that air motion is generated by a source "thrusting forward in like manner the adjoining air, so that sound travels unaltered in quality as far as the disturbance of the air manages to reach." The first mathematical theory of sound propagation was formulated by Isaac Newton (1642–1717), whose ideas included a mechanical interpretation of sound as being "pressure pulses transmitted through neighboring fluid particles." Newton also made a first determination of the speed of sound. A theory of sound propagation resting on firmer mathematical and physical concepts was developed during the eighteenth century by Euler, Lagrange and d'Alembert.

The best-known example of wave motion in a solid is probably provided by the ground motion due to an earthquake, generated by the sliding of tectonic plates over a fault surface, often deep in the interior of the earth. Waves in solids occur, however, in a multitude of other cases of dynamic excitation.

Generally speaking, the high-rate application of a load to a solid body gives rise to wave motion. Waves so induced are used in geophysical prospecting and in quantitative non destructive evaluation. The reflection and scattering of externally excited incident waves can be used for imaging or other techniques of data processing, with as the ultimate goal the detection of inhomogeneities, including defects, or the determination of material properties.

Engineering applications of wave propagation in solids are concerned with the performance of structures under high rates of loading. Other applications are related to high-speed machinery, ultrasonics and piezoelectric phenomena, as well as to such civil engineering practices as pile driving. By now the study of wave propagation effects has become well established in the field of the mechanics of solids.

The study of wave propagation in elastic solids has a long and distinguished history. The early work on elastic waves received its impetus from the view, which was prevalent until the end of the nineteenth century, that light could be regarded as the propagation of a disturbance in an elastic aether. This view was espoused by such great mathematicians as Cauchy and Poisson and to a large extent motivated them to develop what is now generally known as the theory of elasticity. The early investigations on the propagation of waves in elastic solids carried out by Poisson, Ostrogradsky, Cauchy, Green, Lamé, Stokes, Clebsch and Christoffel are discussed in the historical introduction in Love's treatise on the mathematical theory of elasticity (1892).

1.2 Static reciprocity for an elastic body subjected to concentrated loads

In its simplest form an elastostatic reciprocity theorem can be established by considering an elastic body of arbitrary shape. The body is supported in such a way that displacement as a rigid body is impossible. Two states of loading are defined. In the first state the load is P_1, applied at point 1, and in the second state the load is P_2, applied at point 2 (Fig. 1.1). The corresponding displacements at the points of application of the loads and in the direction of load application are δ_{11}, δ_{21}, δ_{22} and δ_{12}. Here δ_{11} is the displacement at point 1 in the direction of P_1 due to the application of P_1, while δ_{21} is the displacement at point 2 in the direction of P_2 due to the application of P_1 at point 1. Analogous definitions hold for δ_{22} and δ_{12}.
The reciprocity relation states that

$$P_1\delta_{12} = P_2\delta_{21}. \tag{1.2.1}$$

For $P_1 = P_2$ we obtain what is known as Maxwell's theorem (1864),

$$\delta_{12} = \delta_{21}. \tag{1.2.2}$$

In words, for forces of equal magnitude P_1 and P_2, the displacement produced by the force P_1 at the point of application of the force P_2 and in the direction

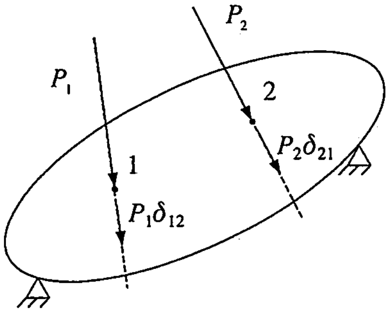

Figure 1.1 Body of volume V and boundary S subjected to concentrated loads P_1 and P_2.

of that force is equal to the displacement produced by the force P_2 at the point of application of the force P_1 and in the direction of P_1.

The usual proof of Eq. (1.2.1) is based on the result that the energy stored in an elastic body after application of the two forces is independent of their sequence of application.

If P_1 is applied first and then P_2, the stored energy is

$$U = \tfrac{1}{2}P_1\delta_{11} + P_1\delta_{12} + \tfrac{1}{2}P_2\delta_{22}. \qquad (1.2.3)$$

However, when P_2 is applied first we have

$$U = \tfrac{1}{2}P_2\delta_{22} + P_2\delta_{21} + \tfrac{1}{2}P_1\delta_{11}. \qquad (1.2.4)$$

Since the stored energy must be independent of the order of application of the two forces, equating the two expressions for U immediately yields Eq. (1.2.1).

1.3 Static reciprocity for distributed body forces and surface loads

The elastostatic result given by Eq. (1.2.1) can be extended to distributions of body forces, surface tractions and displacements. Let us define the two elastostatic states by superscripts A and B:

$$f_i^A(\mathbf{x}), \quad t_i^A(\mathbf{x}) \quad \text{and} \quad u_i^A(\mathbf{x}) \qquad (1.3.1)$$

and

$$f_i^B(\mathbf{x}), \quad t_i^B(\mathbf{x}) \quad \text{and} \quad u_i^B(\mathbf{x}), \qquad (1.3.2)$$

where f_i, t_i and u_i denote the components of body forces, surface tractions and displacements, respectively. The following relationship is derived in detail in Section 2.5:

$$\int_V \left(f_i^A u_i^B - f_i^B u_i^A\right) dV + \int_S \left(t_i^A u_i^B - t_i^B u_i^A\right) dS = 0, \qquad (1.3.3)$$

where S is the boundary of the domain of volume V; see Fig. 1.1. Equation (1.3.3) expresses Betti's reciprocity theorem, one of the most elegant and useful theorems in the linear theory of elasticity.

1.4 The wave equation in one dimension

Every book on wave propagation contains a discussion of the wave equation in one dimension. The equation governs vibrations and wave propagation in many one-dimensional physical systems. Examples are a column of gas, a thin elastic rod and a taut string. The equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad (1.4.1)$$

where c is a velocity (m/s).

A general solution to Eq. (1.4.1) can be obtained by introducing the new variables

$$\alpha = t - x/c, \qquad \beta = t + x/c.$$

Equation (1.4.1) then becomes

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

The general solution of this equation is

$$u = f(\alpha) + g(\beta)$$

or

$$u(x,t) = f(t - x/c) + g(t + x/c). \qquad (1.4.2)$$

This general solution is due to d'Alembert (1747). For the first term the argument remains unchanged if increments in time, Δt, and position, Δx, are related by $\Delta x = c\Delta t$. This implies that, for a position that propagates in the x-direction with velocity c, the argument (or phase) of $f(t - x/c)$, and therefore the magnitude of $f(t - x/c)$, remains unchanged. The function $f(t - x/c)$ is said to represent a wave propagating in the x-direction with phase velocity c.

The shape of the wave does not change as it propagates through the medium, and hence this wave is called non-dispersive. Similarly $g(t + x/c)$ represents a non-dispersive wave propagating in the negative x-direction.

A special case of $u(x, t) = f(t - x/c)$ is a harmonic wave of the form

$$u(x, t) = U \cos[k(ct - x)]. \tag{1.4.3}$$

For fixed t (a snapshot), $u(x, t)$ is a periodic function of x with wavelength $\lambda = 2\pi/k$. Also, U is the amplitude and k is the wavenumber, $k = 2\pi/\lambda$. For fixed x, $u(x, t)$ is a harmonic function of time with circular frequency ω, where

$$\omega = kc. \tag{1.4.4}$$

Equation (1.4.3) may also be written as

$$u(x, t) = U \cos(kx - \omega t). \tag{1.4.5}$$

We define the period as $T = 2\pi/\omega$, and the frequency as $f = 1/T = \omega/(2\pi)$. The frequency, f, is expressed in Hertz (1 Hz = 1 cycle/sec).

The particle velocity follows from (1.4.5) as

$$\dot{u}(x, t) = Ukc \sin[k(x - ct)].$$

Hence

$$(\dot{u}/c)_{\max} = Uk = 2\pi U/\lambda$$

In a linear theory we must have $U/\lambda \ll 1$. Hence

$$\dot{u} \ll c.$$

It is convenient to use complex notation:

$$u(x, t) = u(x)e^{-i\omega t}, \qquad i = \sqrt{-1}, \tag{1.4.6}$$

where

$$u(x) = Ue^{ikx}.$$

It is understood that the physical quantity is either the real or the imaginary part of Eq. (1.4.6).

1.5 Use of a virtual wave in reciprocity considerations

Let us illustrate the use of a virtual wave by a simple one-dimensional example. For this example we consider wave motion on an infinitely long taut string. The governing equation for a string subjected to a distributed load $q(x, t)$ is

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} + \frac{1}{\rho} q(x, t), \qquad \text{where} \qquad c^2 = \frac{T}{\rho} \tag{1.5.1}$$

and $w(x, t)$ is the deflection, ρ the mass density per unit length and T the tension in the string. For the time-harmonic case with time factor $\exp(-i\omega t)$, where $w(x, t)$ is defined analogously to Eq. (1.4.6), Eq. (1.5.1) becomes

$$c^2 \frac{d^2 w}{dx^2} + \omega^2 w + \frac{1}{\rho} q(x) = 0, \tag{1.5.2}$$

where $w \equiv w(x)$ and $q(x)$ depend on x only.

Now let us consider the two states $w^A(x)$ and $w^B(x)$, corresponding to $q^A(x)$ and $q^B(x)$, respectively. When Eq. (1.5.2) is written for states A and B, the equation for state A is multiplied by $w^B(x)$ and the equation for state B is multiplied by $w^A(x)$, and one of the equations is subsequently subtracted from the other, we obtain the local reciprocity relation

$$\frac{d^2 w^A}{dx^2} w^B + \frac{1}{\rho c^2} q^A w^B - \frac{d^2 w^B}{dx^2} w^A - \frac{1}{\rho c^2} q^B w^A = 0. \tag{1.5.3}$$

Integration of Eq. (1.5.3) over x from $x = a$ to $x = b$, using integration by parts, yields the global reciprocity relation

$$\left[\frac{dw^A}{dx} w^B - \frac{dw^B}{dx} w^A \right]_{x=a}^{x=b} + \frac{1}{\rho c^2} \int_a^b \left(q^A w^B - q^B w^A \right) dx = 0. \tag{1.5.4}$$

Now for state A we select the solution to wave motion generated by a concentrated load applied at $x = 0$, i.e.,

$$q^A(x) = P\delta(x), \tag{1.5.5}$$

where $a < x = 0 < b$ and $\delta(x)$ is the Dirac delta function. Because of the symmetry with respect to $x = 0$, and because we know the general form of the solution from the homogeneous form of Eq. (1.5.2) we can write

$$x > 0: \qquad w^A(x) = Re^{ikx}, \tag{1.5.6}$$

$$x < 0: \qquad w^A(x) = Re^{-ikx}, \tag{1.5.7}$$

where it has been taken into account that the time factor is $\exp(-i\omega t)$, R is a radiation constant and

$$k = \frac{\omega}{c}. \tag{1.5.8}$$

For state B we select the following virtual wave:

$$q^B \equiv 0 \qquad \text{and} \qquad w^B = e^{ikx}. \tag{1.5.9}$$

It is noted that, for $x > 0$, $w^A(x)$ and $w^B(x)$ propagate in the same direction, while for $x < 0$ they form a system of counter-propagating waves. Substitution of Eqs. (1.5.5) and (1.5.6)–(1.5.9) into Eq. (1.5.4) yields

$$ikRe^{2ikb} - ikRe^{2ikb} + 2ikR = -\frac{P}{\rho c^2}. \tag{1.5.10}$$

Superposition of the two waves at $x = b$ yields wave forms of the type exp($2ikb$). However, the two terms at $x = b$ cancel. At $x = a$ the superimposed wave forms eliminate each other, yielding constants that add to $2ikR$. This particular kind of interaction between propagating and counter-propagating waves forms the basis for the combined use of reciprocity considerations and a virtual wave. The result shows that we only need to take into account the terms from the side where the actual wave and the virtual wave are counter-propagating.

Equation (1.5.10) has the solution

$$R = \frac{i}{2}\frac{1}{k}\frac{P}{T}. \tag{1.5.11}$$

Thus, we have solved the problem of wave radiation from a concentrated force at $x = 0$ by combining reciprocity considerations with the use of a virtual wave. The problem can be solved in various other ways, one being the use of the exponential Fourier transform with respect to x.

Equation (1.5.4) can also be used conveniently to determine the radiation from a distributed load. Let the distributed load be defined by

$$q^A(x) = f(x) \qquad \text{for} \qquad 0 \le x \le l. \tag{1.5.12}$$

Equation (1.5.10) then becomes

$$2ikR = -\frac{1}{\rho c^2}\int_0^l f(x)e^{ikx}dx. \tag{1.5.13}$$

The expression for $w^A(x)$, Eq. (1.5.6), will be valid for $x > l$ and $x < 0$.

1.6 Synopsis

To save the reader the trouble of looking elsewhere for relevant information on elastodynamic theory, and to establish the notation that is employed throughout the book, Chapter 2 presents a brief discussion of elastodynamic theory. The chapter also includes a summary of equations governing the linear theory of viscoelasticity as well as a statement of the governing equations for an acoustic medium.

Chapter 3 discusses wave motion in an unbounded, homogeneous, isotropic, linearly elastic solid. Plane waves are considered first. The presence of a surface gives rise to reflected waves. The reflection of plane waves incident at an arbitrary angle on a plane surface free of surface tractions is discussed. Expressions for wave motion generated by a point load and a line load are presented, together with their far-field approximations.

Chapters 4, 5 and 6 are concerned with reciprocity in acoustics, in one-dimensional elastodynamics and in three-dimensional elastodynamics, respectively. Each chapter states the pertinent reciprocity relations in the time domain, the frequency domain and the Laplace transform domain. Reciprocity considerations are most easily applied in the frequency domain and the applications are, therefore, primarily concerned with time-harmonic solutions. Simple applications to dynamic problems of the use of reciprocity considerations in conjunction with a virtual wave are presented in these chapters. The purpose is to give the reader a sense of the applicability of reciprocity as a tool for obtaining solutions for acoustic and elastodynamic problems. Most of the examples are very simple. In the literature their solutions are usually obtained by the use of Fourier transform techniques. Some of the cases in Chapter 6, such as the anti-plane elastodynamic field generated by a line load, can be solved in an equally simple or simpler way by a number of other methods. An interesting case is provided by the determination of surface waves on a half-space when an anti-plane line load is applied in its interior and the surface of the half-plane is constrained by an impedance condition. Other cases deal with acoustic wave motion with polar symmetry, reciprocity for elastic waves reflected from a free surface, reciprocity for fields generated by point forces in bounded and unbounded bodies and formulations of scattering and related inverse problems. The chapter ends with a brief summary of examples from the technical literature.

Chapter 7 contains new material for wave motion guided by a carrier wave on a preferred plane. It is shown that the carrier wave satisfies a simple two-dimensional reduced wave equation. Motion away from the plane of the carrier wave is represented by the same expressions, irrespective of the form of the carrier wave. As examples we discuss Rayleigh surface waves and Lamb waves. A few forms of the carrier wave are discussed.

In Chapter 8 we combine this formulation with reciprocity considerations and the use of a virtual wave to determine the surface wave motion of an elastic half-space generated by a sub-surface line load or a point load of arbitrary direction. For the case of the point load, the virtual wave motion that is employed