Elastic Wave Propagation and Generation in Seismology

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Contents

1

Introduction to tensors and dyadics

1.1 Introduction

Tensors play a fundamental role in theoretical physics. The reason for this is that physical laws written in tensor form are independent of the coordinate system used (Morse and Feshbach, 1953). Before elaborating on this point, consider a simple example, based on Segel (1977). Newton's second law is $\mathbf{f} = m\mathbf{a}$, where $\mathbf{f}$ and $\mathbf{a}$ are vectors representing the force and acceleration of an object of mass m. This basic law does not have a coordinate system attached to it. To apply the law in a particular situation it will be convenient to select a coordinate system that simplifies the mathematics, but there is no question that any other system will be equally acceptable. Now consider an example from elasticity, discussed in Chapter 3. The stress vector $\mathbf{T}$ (force/area) across a surface element in an elastic solid is related to the vector $\mathbf{n}$ normal to the same surface via the stress tensor. The derivation of this relation is carried out using a tetrahedron with faces along the three coordinate planes in a Cartesian coordinate system. Therefore, it is reasonable to ask whether the same result would have been obtained if a different Cartesian coordinate system had been used, or if a spherical, or cylindrical, or any other curvilinear system, had been used. Take another example. The elastic wave equation will be derived in a Cartesian coordinate system. As discussed in Chapter 4, two equations will be found, one in component form and one in vector form in terms of a combination of gradient, divergence, and curl. Again, here there are some pertinent questions regarding coordinate systems. For example, can either of the two equations be applied in non-Cartesian coordinate systems? The reader may already know that only the latter equation is generally applicable, but may not be aware that there is a mathematical justification for that fact, namely, that the gradient, divergence, and curl are independent of the coordinate system (Morse and Feshbach, 1953). These questions are generally not discussed in introductory texts, with the consequence that the reader fails to grasp the deeper meaning of the concepts of

1

vectors and tensors. It is only when one realizes that physical entities (such as force, acceleration, stress tensor, and so on) and the relations among them have an existence independent of coordinate systems, that it is possible to appreciate that there is more to tensors than what is usually discussed. It is possible, however, to go through the basic principles of stress and strain without getting into the details of tensor analysis. Therefore, some parts of this chapter are not essential for the rest of the book.

Tensor analysis, in its broadest sense, is concerned with arbitrary curvilinear coordinates. A more restricted approach concentrates on *orthogonal curvilinear coordinates*, such as cylindrical and spherical coordinates. These coordinate systems have the property that the unit vectors at a given point in space are perpendicular (i.e. orthogonal) to each other. Finally, we have the rectangular Cartesian system, which is also orthogonal. The main difference between general orthogonal and Cartesian systems is that in the latter the unit vectors do not change as a function of position, while this is not true in the former. Unit vectors for the spherical system will be given in §9.9.1. The theory of tensors in non-Cartesian systems is exceedingly complicated, and for this reason we will limit our study to Cartesian tensors. However, some of the most important relations will be written using dyadics (see §1.6), which provide a symbolic representation of tensors independent of the coordinate system. It may be useful to note that there are oblique Cartesian coordinate systems of importance in crystallography, for example, but in the following we will consider the rectangular Cartesian systems only.

1.2 Summary of vector analysis

It is assumed that the reader is familiar with the material summarized in this section (see, e.g., Lass, 1950; Davis and Snider, 1991).

A vector is defined as a directed line segment, having both magnitude and direction. The magnitude, or length, of a vector $\mathbf{a}$ will be represented by $|\mathbf{a}|$. The sum and the difference of two vectors, and the multiplication of a vector by a scalar (real number) are defined using geometric rules. Given two vectors $\mathbf{a}$ and $\mathbf{b}$, two products between them have been defined.

Scalar, or dot, product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \alpha, \qquad (1.2.1)$$

where α is the angle between the vectors.

Vector, or cross, product:

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin \alpha) \, \mathbf{n}, \qquad (1.2.2)$$
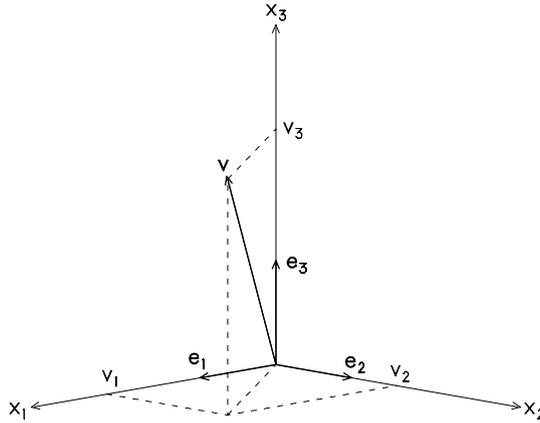
Fig. 1.1. Rectangular Cartesian coordinate system with unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and decomposition of an arbitrary vector $\mathbf{v}$ into components v_1, v_2, v_3.

where α is as before, and $\mathbf{n}$ is a unit vector (its length is equal to 1) perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that the three vectors form a right-handed system.

An important property of the vector product, derived using geometric arguments, is the distributive law

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}. \qquad (1.2.3)$$

By introducing a rectangular Cartesian coordinate system it is possible to write a vector in terms of three components. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ be the three unit vectors along the x_1, x_2, and x_3 axes of Fig. 1.1. Then any vector $\mathbf{v}$ can be written as

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = \sum_{i=1}^{3} v_i\mathbf{e}_i. \qquad (1.2.4)$$

The components v_1, v_2, and v_3 are the orthogonal projections of $\mathbf{v}$ in the directions of the three axes (Fig. 1.1).

Before proceeding, a few words concerning the notation are necessary. A vector will be denoted by a bold-face letter, while its components will be denoted by the same letter in italics with subindices (literal or numerical). A bold-face letter with a subindex represents a vector, not a vector component. The three unit vectors defined above are examples of the latter. If we want to write the kth component of the unit vector $\mathbf{e}_j$ we will write $(\mathbf{e}_j)_k$. For example, $(\mathbf{e}_2)_1 = 0$, $(\mathbf{e}_2)_2 = 1$, and $(\mathbf{e}_2)_3 = 0$. In addition, although vectors will usually be written in row form (e.g., as in (1.2.4)), when they are involved in matrix operations they should be considered as column vectors, i.e., as matrices of one column and three rows. For example, the matrix form of the scalar product $\mathbf{a} \cdot \mathbf{b}$ is $\mathbf{a}^{\mathrm{T}}\mathbf{b}$, where T indicates transposition.

When the scalar product is applied to the unit vectors we find

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0 \tag{1.2.5}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{1.2.6}$$

Equations (1.2.5) and (1.2.6) can be summarized as follows:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{1.2.7}$$

The symbol δ_{jk} is known as the *Kronecker delta*, which is an example of a second-order tensor, and will play an important role in this book. As an example of (1.2.7), $\mathbf{e}_2 \cdot \mathbf{e}_k$ is zero unless $k = 2$, in which case the scalar product is equal to 1.

Next we derive an alternative expression for a vector $\mathbf{v}$. Using (1.2.4), the scalar product of $\mathbf{v}$ and $\mathbf{e}_i$ is

$$\mathbf{v} \cdot \mathbf{e}_i = \left(\sum_{k=1}^{3} v_k \mathbf{e}_k \right) \cdot \mathbf{e}_i = \sum_{k=1}^{3} v_k \mathbf{e}_k \cdot \mathbf{e}_i = \sum_{k=1}^{3} v_k (\mathbf{e}_k \cdot \mathbf{e}_i) = v_i. \tag{1.2.8}$$

Note that when applying (1.2.4) the subindex in the summation must be different from i. To obtain (1.2.8) the following were used: the distributive law of the scalar product, the law of the product by a scalar, and (1.2.7). Equation (1.2.8) shows that the ith component of $\mathbf{v}$ can be written as

$$v_i = \mathbf{v} \cdot \mathbf{e}_i. \tag{1.2.9}$$

When (1.2.9) is introduced in (1.2.4) we find

$$\mathbf{v} = \sum_{i=1}^{3} (\mathbf{v} \cdot \mathbf{e}_i) \mathbf{e}_i. \tag{1.2.10}$$

This expression will be used in the discussion of dyadics (see §1.6).

In terms of its components the length of the vector is given by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = (\mathbf{v} \cdot \mathbf{v})^{1/2}. \tag{1.2.11}$$

Using purely geometric arguments it is found that the scalar and vector products can be written in component form as follows:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{1.2.12}$$

and

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{e}_1 + (u_3 v_1 - u_1 v_3)\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\mathbf{e}_3. \tag{1.2.13}$$

The last expression is based on the use of (1.2.3).

Vectors, and vector operations such as the scalar and vector products, among others, are defined independently of any coordinate system. Vector relations derived without recourse to vector components will be valid when written in component form regardless of the coordinate system used. Of course, the same vector may (and generally will) have different components in different coordinate systems, but they will represent the same geometric entity. This is true for Cartesian and more general coordinate systems, such as spherical and cylindrical ones, but in the following we will consider the former only.

Now suppose that we want to define new vector entities based on operations on the components of other vectors. In view of the comments in §1.1 it is reasonable to expect that not every arbitrary definition will represent a vector, i.e., an entity intrinsically independent of the coordinate system used to represent the space. To see this consider the following example, which for simplicity refers to vectors in two-dimensional (2-D) space. Given a vector $\mathbf{u} = (u_1, u_2)$, define a new vector $\mathbf{v} = (u_1 + \lambda, u_2 + \lambda)$, where λ is a nonzero scalar. Does this definition result in a vector? To answer this question draw the vectors $\mathbf{u}$ and $\mathbf{v}$ (Fig. 1.2a), rotate the original coordinate axes, decompose $\mathbf{u}$ into its new components u_1' and u_2', add λ to each of them, and draw the new vector $\mathbf{v}' = (u_1' + \lambda, u_2' + \lambda)$. Clearly, $\mathbf{v}$ and $\mathbf{v}'$ are not the same geometric object. Therefore, our definition does not represent a vector.

Now consider the following definition: $\mathbf{v} = (\lambda u_1, \lambda u_2)$. After a rotation similar to the previous one we see that $\mathbf{v} = \mathbf{v}'$ (Fig. 1.2b), which is not surprising, as this definition corresponds to the multiplication of a vector by a scalar.

Let us look now at a more complicated example. Suppose that given two vectors $\mathbf{u}$ and $\mathbf{v}$ we want to define a third vector $\mathbf{w}$ as follows:

$$\mathbf{w} = (u_2 v_3 + u_3 v_2)\mathbf{e}_1 + (u_3 v_1 + u_1 v_3)\mathbf{e}_2 + (u_1 v_2 + u_2 v_1)\mathbf{e}_3. \qquad (1.2.14)$$

Note that the only difference with the vector product (see (1.2.13)) is the replacement of the minus signs by plus signs. As before, the question is whether this definition is independent of the coordinate system. In this case, however, finding an answer is not straightforward. What one should do is to compute the components w_1, w_2, w_3 in the original coordinate system, draw $\mathbf{w}$, perform a rotation of axes, find the new components of $\mathbf{u}$ and $\mathbf{v}$, compute w_1', w_2', and w_3', draw $\mathbf{w}'$ and compare it with $\mathbf{w}$. If it is found that the two vectors are different, then it is obvious that (1.2.14) does not define a vector. If the two vectors are equal it might be tempting to say that (1.2.14) does indeed define a vector, but this conclusion would not be correct because there may be other rotations for which $\mathbf{w}$ and $\mathbf{w}'$ are not equal.

These examples should convince the reader that establishing the vectorial character of an entity defined by its components requires a definition of a vector that will take this question into account automatically. Only then will it be possible to
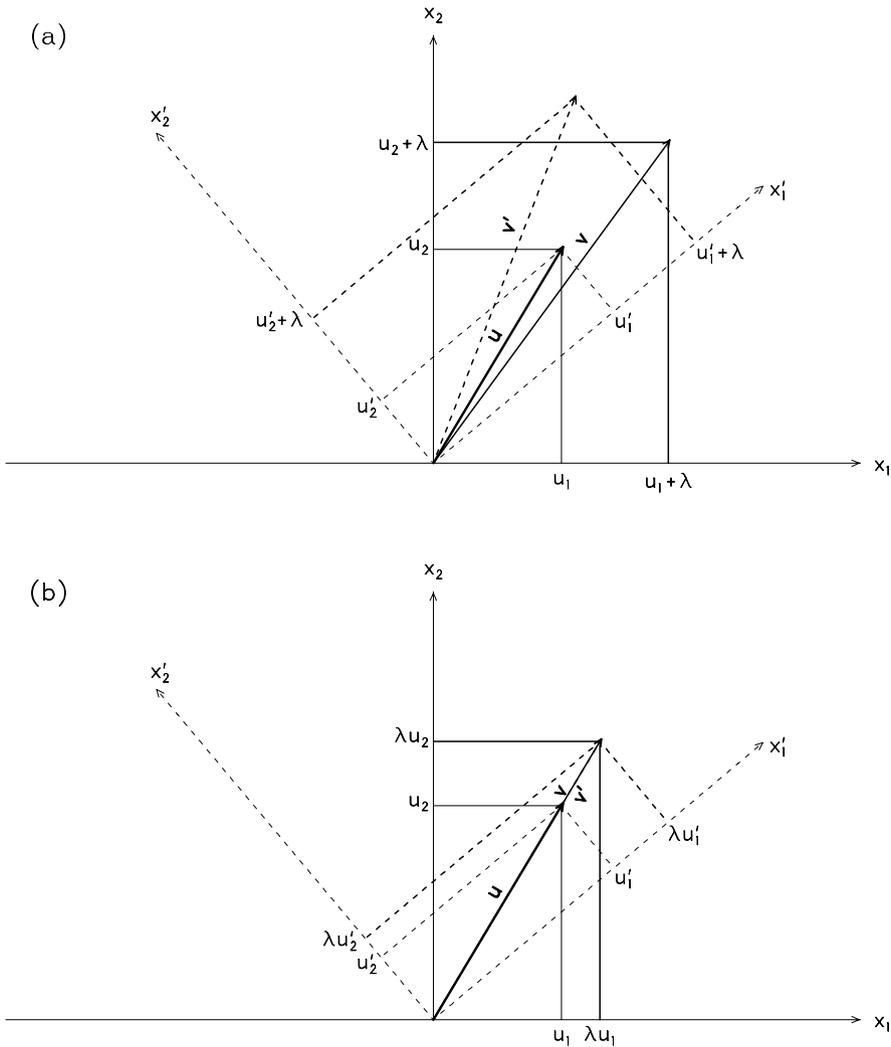
Fig. 1.2. (a) Vectors **v** and **v**′ obtained from a vector **u** as follows. For **v**, add a constant λ to the components u_1 and u_2. For **v**′, add a constant λ to the components u_1' and u_2' obtained by rotation of the axis. Because **v** and **v**′ are not the same vector, we can conclude that the entity obtained by adding a constant to the components of a vector does constitute a vector under a rotation of coordinates. (b) Similar to the construction above, but with the constant λ multiplying the vector components. In this case **v** and **v**′ coincide, which agrees with the fact that the operation defined is just the multiplication of a vector by a scalar. After Santalo (1969).

answer the previous question in a general way. However, before introducing the new definition it is necessary to study coordinate rotations in some more detail. This is done next.

1.3 Rotation of Cartesian coordinates. Definition of a vector

Let Ox_1, Ox_2, and Ox_3 represent a Cartesian coordinate system and Ox'_1, Ox'_2, Ox'_3 another system obtained from the previous one by a rotation about their common origin O (Fig. 1.3). Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ and $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3$ be the unit vectors along the three axes in the original and rotated systems. Finally, let a_{ij} denote the cosine of the angle between Ox'_i and Ox_j. The a_{ij}'s are known as direction cosines, and are related to $\mathbf{e}'_i$ and $\mathbf{e}_j$ by

$$\mathbf{e}'_i \cdot \mathbf{e}_j = a_{ij}. \tag{1.3.1}$$

Given an arbitrary vector $\mathbf{v}$ with components v_1, v_2, and v_3 in the original system, we are interested in finding the components v'_1, v'_2, and v'_3 in the rotated system. To find the relation between the two sets of components we will consider first the relation between the corresponding unit vectors. Using (1.3.1) $\mathbf{e}'_i$ can be written as

$$\mathbf{e}'_i = a_{i1}\mathbf{e}_1 + a_{i2}\mathbf{e}_2 + a_{i3}\mathbf{e}_3 = \sum_{j=1}^{3} a_{ij}\mathbf{e}_j \tag{1.3.2}$$

(Problem 1.3a).
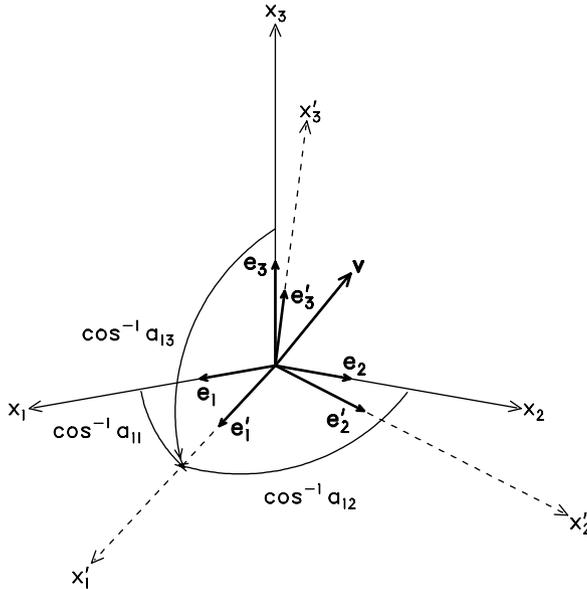
Fig. 1.3. Rotation of axes. Primed and unprimed quantities refer to the original and rotated coordinate systems, respectively. Both systems are rectangular Cartesian. The quantities a_{ij} indicate the scalar product $\mathbf{e}'_i \cdot \mathbf{e}_j$. The vector $\mathbf{v}$ exists independent of the coordinate system. Three relevant angles are shown.

Furthermore, in the original and rotated systems **v** can be written as

$$\mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e}_j \tag{1.3.3}$$

and

$$\mathbf{v} = \sum_{i=1}^{3} v_i' \mathbf{e}_i'. \tag{1.3.4}$$

Now introduce (1.3.2) in (1.3.4)

$$\mathbf{v} = \sum_{i=1}^{3} v_i' \sum_{j=1}^{3} a_{ij} \mathbf{e}_j \equiv \sum_{j=1}^{3} \left(\sum_{i=1}^{3} a_{ij} v_i' \right) \mathbf{e}_j. \tag{1.3.5}$$

Since (1.3.3) and (1.3.5) represent the same vector, and the three unit vectors $\mathbf{e}_1$, $\mathbf{e}_1$, and $\mathbf{e}_3$ are independent of each other, we conclude that

$$v_j = \sum_{i=1}^{3} a_{ij} v_i'. \tag{1.3.6}$$

If we write the $\mathbf{e}_j$s in terms of the $\mathbf{e}_i'$s and replace them in (1.3.3) we find that

$$v_i' = \sum_{j=1}^{3} a_{ij} v_j \tag{1.3.7}$$

(Problem 1.3b).

Note that in (1.3.6) the sum is over the first subindex of a_{ij}, while in (1.3.7) the sum is over the second subindex of a_{ij}. This distinction is critical and must be respected.

Now we are ready to introduce the following definition of a vector:

three scalars are the components of a vector if under a rotation of coordinates they transform according to (1.3.7).

What this definition means is that if we want to define a vector by some set of rules, we have to verify that the vector components satisfy the transformation equations.

Before proceeding we will introduce a *summation convention* (due to Einstein) that will simplify the mathematical manipulations significantly. The convention applies to monomial expressions (such as a single term in an equation) and consists of dropping the sum symbol and summing over repeated indices.[1] This convention requires that the same index should appear no more than twice in the same term.

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[1] In this book the convention will not be applied to uppercase indices

Repeated indices are known as *dummy indices*, while those that are not repeated are called *free indices*. Using this convention, we will write, for example,

$$\mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e}_j = v_j \mathbf{e}_j \tag{1.3.8}$$

$$v_j = \sum_{i=1}^{3} a_{ij} v_i' = a_{ij} v_i' \tag{1.3.9}$$

$$v_i' = \sum_{j=1}^{3} a_{ij} v_j = a_{ij} v_j. \tag{1.3.10}$$

It is important to have a clear idea of the difference between free and dummy indices. A particular dummy index can be changed at will as long as it is replaced (in its two occurrences) by some other index not equal to any other existing indices in the same term. Free indices, on the other hand, are fixed and cannot be changed inside a single term. However, a free index can be replaced by another as long as the change is effected in all the terms in an equation, and the new index is different from all the other indices in the equation. In (1.3.9) i is a dummy index and j is a free index, while in (1.3.10) their role is reversed. The examples below show legal and illegal index manipulations.

The following relations, derived from (1.3.9), are true

$$v_j = a_{ij} v_i' = a_{kj} v_k' = a_{lj} v_l' \tag{1.3.11}$$

because the repeated index i was replaced by a different repeated index (equal to k or l). However, it would not be correct to replace i by j because j is already present in the equation. If i were replaced by j we would have

$$v_j = a_{jj} v_j', \tag{1.3.12}$$

which would not be correct because the index j appears more than twice in the right-hand term, which is not allowed. Neither would it be correct to write

$$v_j = a_{ik} v_i' \tag{1.3.13}$$

because the free index j has been changed to k only in the right-hand term. On the other hand, (1.3.9) can be written as

$$v_k = a_{ik} v_i' \tag{1.3.14}$$

because the free index j has been replaced by k on both sides of the equation.

As (1.3.9) and (1.3.10) are of fundamental importance, it is necessary to pay attention to the fact that in the former the sum is over the first index of a_{ij} while

in the latter the sum is over the second index of a_{ij}. Also note that (1.3.10) can be written as the product of a matrix and a vector:

$$\mathbf{v}' = \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \equiv \mathbf{A}\,\mathbf{v}, \qquad (1.3.15)$$

where $\mathbf{A}$ is the matrix with elements a_{ij}.

It is clear that (1.3.9) can be written as

$$\mathbf{v} = \mathbf{A}^{\mathrm{T}}\mathbf{v}', \qquad (1.3.16)$$

where the superscript T indicates transposition.

Now we will derive an important property of $\mathbf{A}$. By introducing (1.3.10) in (1.3.9) we obtain

$$v_j = a_{ij} a_{ik} v_k. \qquad (1.3.17)$$

Note that it was necessary to change the dummy index in (1.3.10) to satisfy the summation convention. Equation (1.3.17) implies that any of the three components of $\mathbf{v}$ is a combination of all three components. However, this cannot be generally true because $\mathbf{v}$ is an arbitrary vector. Therefore, the right-hand side of (1.3.17) must be equal to v_j, which in turn implies that the product $a_{ij} a_{ik}$ must be equal to unity when $j = k$, and equal to zero when $j \neq k$. This happens to be the definition of the Kronecker delta δ_{jk} introduced in (1.2.7), so that

$$a_{ij} a_{ik} = \delta_{jk}. \qquad (1.3.18)$$

If (1.3.9) is introduced in (1.3.10) we obtain

$$a_{ij} a_{kj} = \delta_{ik}. \qquad (1.3.19)$$

Setting $i = k$ in (1.3.19) and writing in full gives

$$1 = a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = |\mathbf{e}_i'|^2; \qquad i = 1, 2, 3, \qquad (1.3.20)$$

where the equality on the right-hand side follows from (1.3.2).

When $i \neq k$, (1.3.19) gives

$$0 = a_{i1} a_{k1} + a_{i2} a_{k2} + a_{i3} a_{k3} = \mathbf{e}_i' \cdot \mathbf{e}_k', \qquad (1.3.21)$$

where the equality on the right-hand side also follows from (1.3.2). Therefore, (1.3.19) summarizes the fact that the $\mathbf{e}_j'$s are unit vectors orthogonal to each other, while (1.3.18) does the same thing for the $\mathbf{e}_i$s. Any set of vectors having these properties is known as an orthonormal set.

In matrix form, (1.3.18) and (1.3.19) can be written as

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}, \tag{1.3.22}$$

where $\mathbf{I}$ is the identity matrix.

Equation (1.3.22) can be rewritten in the following useful way:

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}; \qquad (\mathbf{A}^{\mathrm{T}})^{-1} = \mathbf{A}, \tag{1.3.23}$$

where the superscript -1 indicates matrix inversion. From (1.3.22) we also find

$$|\mathbf{A}\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}||\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|^2 = |\mathbf{I}| = 1, \tag{1.3.24}$$

where vertical bars indicate the determinant of a matrix.

Linear transformations with a matrix such that its determinant squared is equal to 1 are known as orthogonal transformations. When $|\mathbf{A}| = 1$, the transformation corresponds to a rotation. When $|\mathbf{A}| = -1$, the transformation involves the reflection of one coordinate axis in a coordinate plane. An example of reflection is the transformation that leaves the x_1 and x_2 axes unchanged and replaces the x_3 axis by $-x_3$. Reflections change the orientation of the space: if the original system is right-handed, then the new system is left-handed, and vice versa.

1.4   Cartesian tensors

In subsequent chapters the following three tensors will be introduced.

(1) The strain tensor ε_{ij}:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \qquad i, j = 1, 2, 3, \tag{1.4.1}$$

where the vector $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement suffered by a particle inside a body when it is deformed.

(2) The stress tensor τ_{ij}:

$$T_i = \tau_{ij}n_j; \qquad i = 1, 2, 3, \tag{1.4.2}$$

where T_i and n_j indicate the components of the stress vector and normal vector referred to in §1.1.

(3) The elastic tensor c_{ijkl}, which relates stress to strain:

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl}. \tag{1.4.3}$$

Let us list some of the differences between vectors and tensors. First, while a vector can be represented by a single symbol, such as $\mathbf{u}$, or by its components, such as u_j, a tensor can only be represented by its components (e.g., ε_{ij}), although the introduction of dyadics (see §1.6) will allow the representation of tensors by

single symbols. Secondly, while vector components carry only one subindex, tensors carry two subindices or more. Thirdly, in the three-dimensional space we are considering, a vector has three components, while ε_{ij} and τ_{ij} have 3×3, or nine, components, and c_{ijkl} has 81 components ($3 \times 3 \times 3 \times 3$). Tensors ε_{ij} and τ_{ij} are known as second-order tensors, while c_{ijkl} is a fourth-order tensor, with the order of the tensor being given by the number of free indices. There are also differences among the tensors shown above. For example, ε_{ij} is defined in terms of operations (derivatives) on the components of a single vector, while τ_{ij} appears in a relation between two vectors. c_{ijkl}, on the other hand, relates two tensors.

Clearly, tensors offer more variety than vectors, and because they are defined in terms of components, the comments made in connection with vector components and the rotation of axes also apply to tensors. To motivate the following definition of a second-order tensor consider the relation represented by (1.4.2). For this relation to be independent of the coordinate system, upon a rotation of axes we must have

$$T'_l = \tau'_{lk} n'_k. \tag{1.4.4}$$

In other words, the functional form of the relation must remain the same after a change of coordinates. We want to find the relation between τ'_{lk} and τ_{ij} that satisfies (1.4.2) and (1.4.4). To do that multiply (1.4.2) by a_{li} and sum over i:

$$a_{li} T_i = a_{li} \tau_{ij} n_j. \tag{1.4.5}$$

Before proceeding rewrite T'_l and n_j using (1.3.10) with $\mathbf{v}$ replaced by $\mathbf{T}$ and (1.3.9) with $\mathbf{v}$ replaced by $\mathbf{n}$. This gives

$$T'_l = a_{li} T_i; \qquad n_j = a_{kj} n'_k. \tag{1.4.6a,b}$$

From (1.4.6a,b), (1.4.2), and (1.4.5) we find

$$T'_l = a_{li} T_i = a_{li} \tau_{ij} n_j = a_{li} \tau_{ij} a_{kj} n'_k = (a_{li} a_{kj} \tau_{ij}) n'_k. \tag{1.4.7}$$

Now subtracting (1.4.4) from (1.4.7) gives

$$0 = (\tau'_{lk} - a_{li} a_{kj} \tau_{ij}) n'_k. \tag{1.4.8}$$

As n_k is an arbitrary vector, the factor in parentheses in (1.4.8) must be equal to zero (Problem 1.7), so that

$$\tau'_{lk} = a_{li} a_{kj} \tau_{ij}. \tag{1.4.9}$$

Note that (1.4.9) does not depend on the physical nature of the quantities involved in (1.4.2). Only the functional relation matters. This result motivates the following definition.

Second-order tensor. Given nine quantities t_{ij}, they constitute the components of a second-order tensor if they transform according to

$$t'_{ij} = a_{il}a_{jk}t_{lk} \tag{1.4.10}$$

under a change of coordinates $v'_i = a_{ij}v_j$.

To write the tensor components in the unprimed system in terms of the components in the primed system, multiply (1.4.10) by $a_{im}a_{jn}$ and sum over i and j, and use the orthogonality relation (1.3.18):

$$a_{im}a_{jn}t'_{ij} = a_{im}a_{jn}a_{il}a_{jk}t_{lk} = a_{im}a_{il}a_{jn}a_{jk}t_{lk} = \delta_{lm}\delta_{kn}t_{lk} = t_{mn}. \tag{1.4.11}$$

Therefore,

$$t_{mn} = a_{im}a_{jn}t'_{ij}. \tag{1.4.12}$$

As (1.4.10) and (1.4.12) are similar, it is important to make sure that the arrangement of indices is strictly adhered to.

Equation (1.4.11) illustrates an important aspect of the Kronecker delta. For a given m and n, the expression $\delta_{lm}\delta_{kn}t_{lk}$ is a double sum over l and k, so that there are nine terms. However, since δ_{lm} and δ_{kn} are equal to zero except when $l = m$ and $k = n$, in which case the deltas are equal to one, the only nonzero term in the sum is t_{mn}. Therefore, the equality $\delta_{lm}\delta_{kn}t_{lk} = t_{mn}$ can be derived formally by replacing l and k in t_{lk} by m and n and by dropping the deltas.

The extension of (1.4.10) to higher-order tensors is straightforward.

Tensor of order n. Given 3^n quantities $t_{i_1 i_2 \ldots i_n}$, they constitute the components of a tensor of order n if they transform according to

$$t'_{i_1 i_2 \ldots i_n} = a_{i_1 j_1}a_{i_2 j_2} \ldots a_{i_n j_n}t_{j_1 j_2 \ldots j_n} \tag{1.4.13}$$

under a change of coordinates $v'_i = a_{ij}v_j$. All the indices $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$ can be 1, 2, 3.

The extension of (1.4.12) to tensors of order n is obvious. For example, for a third-order tensor we have the following relations:

$$t'_{ijk} = a_{il}a_{jm}a_{kn}t_{lmn}; \qquad t_{mnp} = a_{im}a_{jn}a_{kp}t'_{ijk}. \tag{1.4.14a,b}$$

The definition of tensors can be extended to include vectors and scalars, which can be considered as tensors of orders one and zero, respectively. The corresponding number of free indices are one and zero, but this does not mean that dummy indices are not allowed (such as in t_{jj} and $u_{i,i}$, introduced below).

An important consequence of definitions (1.4.10) and (1.4.13) is that if the components of a tensor are all equal to zero in a given coordinate system, they will be equal to zero in any other coordinate system.

In the following we will indicate vectors and tensors by their individual components. For example, u_i, u_j, and u_m, etc., represent the same vector $\mathbf{u}$, while t_{ij}, t_{ik}, t_{mn}, etc., represent the same tensor. As noted earlier, the introduction of dyadics will afford the representation of tensors by single symbols, but for the special case of a second-order tensor we can associate it with a 3×3 matrix. For example, the matrix $\mathbf{T}$ corresponding to the tensor t_{ij} is given by

$$\mathbf{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}. \tag{1.4.15}$$

Introduction of $\mathbf{T}$ is convenient because (1.4.10), which can be written as

$$t'_{ij} = a_{il}t_{lk}a_{jk}, \tag{1.4.16}$$

has a simple expression in matrix form:

$$\mathbf{T}' = \mathbf{ATA}^{\mathrm{T}}, \tag{1.4.17}$$

where $\mathbf{A}$ is the matrix introduced in (1.3.15) and

$$\mathbf{T}' = \begin{pmatrix} t'_{11} & t'_{12} & t'_{13} \\ t'_{21} & t'_{22} & t'_{23} \\ t'_{31} & t'_{32} & t'_{33} \end{pmatrix}. \tag{1.4.18}$$

Equation (1.4.17) is very useful in actual computations.

To express $\mathbf{T}$ in terms of $\mathbf{T}'$, write (1.4.12) as

$$t_{mn} = a_{im}t'_{ij}a_{jn}, \tag{1.4.19}$$

which gives

$$\mathbf{T} = \mathbf{A}^{\mathrm{T}}\mathbf{T}'\mathbf{A}. \tag{1.4.20}$$

Otherwise, solve (1.4.17) for $\mathbf{T}$ by multiplying both sides by $(\mathbf{A}^{\mathrm{T}})^{-1}$ on the right and by $\mathbf{A}^{-1}$ on the left and use (1.3.23).

1.4.1 Tensor operations

(1) *Addition or subtraction of two tensors.* The result is a new tensor whose components are the sum or difference of the corresponding components. These two operations are defined for tensors of the same order. As an example, given the tensors t_{ij} and s_{ij}, their sum or difference is the tensor b_{ij} with components

$$b_{ij} = t_{ij} \pm s_{ij}. \tag{1.4.21}$$

Let us verify that b_{ij} is indeed a tensor. This requires writing the components

of t_{ij} and s_{ij} in the primed system, adding or subtracting them together and verifying that (1.4.10) is satisfied. From (1.4.10) we obtain

$$t'_{ij} = a_{il}a_{jm}t_{lm} \tag{1.4.22}$$

$$s'_{ij} = a_{il}a_{jm}s_{lm}. \tag{1.4.23}$$

Adding or subtracting (1.4.22) and (1.4.23) gives

$$b'_{ij} = t'_{ij} \pm s'_{ij} = a_{il}a_{jm}(t_{lm} \pm s_{lm}) = a_{il}a_{jm}b_{lm}. \tag{1.4.24}$$

Equation (1.4.24) shows that b_{ij} transforms according to (1.4.10) and is therefore a tensor.

(2) *Multiplication of a tensor by a scalar.* Each component of the tensor is multiplied by the scalar. The result of this operation is a new tensor. For example, multiplication of t_{ij} by a scalar λ gives the tensor b_{ij} with components

$$b_{ij} = \lambda t_{ij}. \tag{1.4.25}$$

To show that b_{ij} is a tensor we proceed as before

$$b'_{ij} = \lambda t'_{ij} = \lambda a_{il}a_{jm}t_{lm} = a_{il}a_{jm}(\lambda t_{lm}) = a_{il}a_{jm}b_{lm}. \tag{1.4.26}$$

(3) *Outer product of two tensors.* This gives a new tensor whose order is equal to the sum of the orders of the two tensors and whose components are obtained by multiplication of the components of the two tensors. From this definition it should be clear that the indices in one tensor cannot appear among the indices in the other tensor. As an example, consider the outer product of t_{ij} and u_k. The result is the tensor s_{ijk} given by

$$s_{ijk} = t_{ij}u_k. \tag{1.4.27}$$

To show that s_{ijk} is a tensor proceed as before

$$s'_{ijk} = t'_{ij}u'_k = a_{il}a_{jm}t_{lm}a_{kn}u_n = a_{il}a_{jm}a_{kn}t_{lm}u_n = a_{il}a_{jm}a_{kn}s_{lmn}. \tag{1.4.28}$$

As another example, the outer product of two vectors **a** and **b** is the tensor with components a_ib_j. This particular outer product will be considered again when discussing dyadics.

(4) *Contraction of indices.* In a tensor of order two or higher, set two indices equal to each other. As the two contracted indices become dummy indices, they indicate a sum over the repeated index. By contraction, the order of the tensor is reduced by two. For example, given t_{ij}, by contraction of i and j we obtain the scalar

$$t_{ii} = t_{11} + t_{22} + t_{33}, \tag{1.4.29}$$

which is known as the *trace* of t_{ij}. Note that when t_{ij} is represented by a matrix,

t_{ii} corresponds to the sum of its diagonal elements, which is generally known as the trace of the matrix. Another example of contraction is the divergence of a vector, discussed in §1.4.5.

(5) *Inner product, or contraction, of two tensors.* Given two tensors, first form their outer product and then apply a contraction of indices using one index from each tensor. For example, the scalar product of two vectors **a** and **b** is equal to their inner product, given by $a_i b_i$. We will refer to the inner product as the contraction of two tensors. By extension, a product involving a_{ij}, as in (1.4.5) and (1.4.11), for example, will also be called a contraction.

1.4.2 Symmetric and anti-symmetric tensors

A second-order tensor t_{ij} is symmetric if

$$t_{ij} = t_{ji} \tag{1.4.30}$$

and anti-symmetric if

$$t_{ij} = -t_{ji}. \tag{1.4.31}$$

Any second-order tensor b_{ij} can be written as the following identity:

$$b_{ij} \equiv \frac{1}{2}b_{ij} + \frac{1}{2}b_{ij} + \frac{1}{2}b_{ji} - \frac{1}{2}b_{ji} = \frac{1}{2}(b_{ij} + b_{ji}) + \frac{1}{2}(b_{ij} - b_{ji}). \tag{1.4.32}$$

Clearly, the tensors in parentheses are symmetric and anti-symmetric, respectively. Therefore, b_{ij} can be written as

$$b_{ij} = s_{ij} + a_{ij} \tag{1.4.33}$$

with

$$s_{ij} = s_{ji} = \frac{1}{2}(b_{ij} + b_{ji}) \tag{1.4.34}$$

and

$$a_{ij} = -a_{ji} = \frac{1}{2}(b_{ij} - b_{ji}). \tag{1.4.35}$$

Examples of symmetric second-order tensors are the Kronecker delta, the strain tensor (as can be seen from (1.4.1)), and the stress tensor, as will be shown in the next chapter.

For higher-order tensors, symmetry and anti-symmetry are referred to pairs of indices. A tensor is completely symmetric (anti-symmetric) if it is symmetric (anti-symmetric) for all pairs of indices. For example, if t_{ijk} is completely symmetric, then

$$t_{ijk} = t_{jik} = t_{ikj} = t_{kji} = t_{kij} = t_{jki}. \tag{1.4.36}$$

If t_{ijk} is completely anti-symmetric, then

$$t_{ijk} = -t_{jik} = -t_{ikj} = t_{kij} = -t_{kji} = t_{jki}. \qquad (1.4.37)$$

The permutation symbol, introduced in §1.4.4, is an example of a completely anti-symmetric entity.

The elastic tensor in (1.4.3) has symmetry properties different from those described above. They will be described in detail in Chapter 4.

1.4.3 Differentiation of tensors

Let t_{ij} be a function of the coordinates x_i ($i = 1, 2, 3$). From (1.4.10) we know that

$$t'_{ij} = a_{ik}a_{jl}t_{kl}. \qquad (1.4.38)$$

Now differentiate both sides of (1.4.38) with respect to x'_m,

$$\frac{\partial t'_{ij}}{\partial x'_m} = a_{ik}a_{jl}\frac{\partial t_{kl}}{\partial x_s}\frac{\partial x_s}{\partial x'_m}. \qquad (1.4.39)$$

Note that on the right-hand side we used the chain rule of differentiation and that there is an implied sum over the index s. Also note that since

$$x_s = a_{ms}x'_m \qquad (1.4.40)$$

(Problem 1.8) then

$$\frac{\partial x_s}{\partial x'_m} = a_{ms}. \qquad (1.4.41)$$

Using (1.4.41) and introducing the notation

$$\frac{\partial t'_{ij}}{\partial x'_m} \equiv t'_{ij,m}; \qquad \frac{\partial t_{kl}}{\partial x_s} \equiv t_{kl,s} \qquad (1.4.42)$$

equation (1.4.39) becomes

$$t'_{ij,m} = a_{ik}a_{jl}a_{ms}t_{kl,s}. \qquad (1.4.43)$$

This shows that $t_{kl,s}$ is a third-order tensor.

Applying the same arguments to higher-order tensors shows that first-order differentiation generates a new tensor with the order increased by one. It must be emphasized, however, that in general curvilinear coordinates this differentiation does not generate a tensor.

1.4.4 The permutation symbol

This is indicated by ϵ_{ijk} and is defined by

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any two indices are repeated} \\ 1, & \text{if } ijk \text{ is an even permutation of 123} \\ -1, & \text{if } ijk \text{ is an odd permutation of 123.} \end{cases} \qquad (1.4.44)$$

A permutation is even (odd) if the number of exchanges of i, j, k required to order them as 123 is even (odd). For example, to go from 213 to 123 only one exchange is needed, so the permutation is odd. On the other hand, to go from 231 to 123, two exchanges are needed: $231 \rightarrow 213 \rightarrow 123$, and the permutation is even. After considering all the possible combinations we find

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \qquad (1.4.45)$$

$$\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1. \qquad (1.4.46)$$

The permutation symbol is also known as the alternating or Levi-Civita symbol. The definition (1.4.44) is general in the sense that it can be extended to more than three subindices. The following equivalent definition can only be used with three subindices, but is more convenient for practical uses:

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any two indices are repeated} \\ 1, & \text{if } ijk \text{ are in cyclic order} \\ -1, & \text{if } ijk \text{ are not in cyclic order.} \end{cases} \qquad (1.4.47)$$

Three different indices ijk are in cyclic order if they are equal to 123 or 231 or 312. They are not in cyclic order if they are equal to 132 or 321 or 213 (see Fig. 1.4).

The permutation symbol will play an important role in the applications below, so its properties must be understood well. When using ϵ_{ijk} in equations, the values of ijk are generally not specified. Moreover, it may happen that the same equation includes factors such as ϵ_{jik} and ϵ_{kij}. In cases like that it is necessary to express
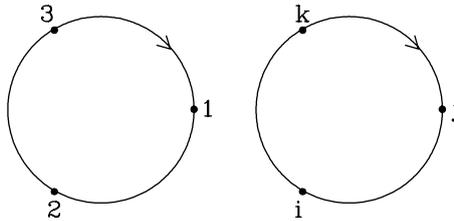
Fig. 1.4. Diagrams used to find out whether or not a combination of the integers 1, 2, 3, or any three indices i, j, k, are in cyclic order (indicated by the arrow). For example, the combination 312 is in cyclic order, while 213 is not. For arbitrary indices i, j, k, if the combination ikj is assumed to be in cyclic order, then combinations such as kji and jik will also be in cyclic order, but kij and ijk will not.

one of them in terms of the other. To do that assume that jik is in cyclic order (Fig. 1.4) and by inspection find out whether kij is in the same order or not. As it is not, $\epsilon_{kij} = -\epsilon_{jik}$.

Is the permutation symbol a tensor? Almost. If the determinant of the transformation matrix $\mathbf{A}$ is equal to 1, then the components of ϵ_{ijk} transform according to (1.4.14a). However, if the determinant is equal to -1, the components of ϵ_{ijk} transform according to -1 times the right-hand side of (1.4.14a). Entities with this type of transformation law are known as *pseudo tensors*. Another well-known example is the vector product, which produces a *pseudo vector*, rather than a vector.

Another important aspect of ϵ_{ijk} is that its components are independent of the coordinate system (see §1.4.7). A proof of the tensorial character of ϵ_{ijk} can be found in Goodbody (1982) and McConnell (1957).

1.4.5 Applications and examples

In the following it will be assumed that the scalars, vectors, and tensors are functions of x_1, x_2, and x_3, as needed, and that the required derivatives exist.

(1) By contraction of the second-order tensor $u_{i,j}$ we obtain the scalar $u_{i,i}$. When this expression is written in full it is clear that it corresponds to the *divergence* of $\mathbf{u}$:

$$u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{div } \mathbf{u} = \nabla \cdot \mathbf{u}. \tag{1.4.48}$$

In the last term we introduced the vector operator ∇ (nabla or del), which can be written as

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right). \tag{1.4.49}$$

This definition of ∇ is valid in Cartesian coordinate systems only.

(2) The derivative of the scalar function $f(x_1, x_2, x_3)$ with respect to x_i is the ith component of the *gradient* of f:

$$\frac{\partial f}{\partial x_i} = (\nabla f)_i = f_{,i}. \tag{1.4.50}$$

(3) The sum of second derivatives $f_{,ii}$ of a scalar function f is the *Laplacian* of f:

$$f_{,ii} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = \nabla^2 f. \tag{1.4.51}$$

(4) The second derivatives of the vector components u_i are $u_{i,jk}$. By contraction we obtain $u_{i,jj}$, which corresponds to the ith component of the Laplacian of the vector $\mathbf{u}$:

$$\nabla^2 \mathbf{u} = \left(\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3\right) = \left(u_{1,jj}, u_{2,jj}, u_{3,jj}\right). \tag{1.4.52}$$

Again, this definition applies to Cartesian coordinates only. In general orthogonal coordinate systems the *Laplacian of a vector* is defined by

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}, \tag{1.4.53}$$

where the appropriate expressions for the gradient, divergence, and curl should be used (e.g., Morse and Feshbach, 1953; Ben-Menahem and Singh, 1981). For Cartesian coordinates (1.4.52) and (1.4.53) lead to the same expression (Problem 1.9).

(5) Show that the Kronecker delta is a second-order tensor. Let us apply the transformation law (1.4.10):

$$\delta'_{ij} = a_{il}a_{jk}\delta_{lk} = a_{il}a_{jl} = \delta_{ij}. \tag{1.4.54}$$

This shows that δ_{ij} is a tensor. Note that in the term to the right of the first equality there is a double sum over l and k (nine terms) but because of the definition of the delta, the only nonzero terms are those with $l = k$ (three terms), in which case delta has a value of one. The last equality comes from (1.3.19).

(6) Let $\mathbf{B}$ be the 3×3 matrix with elements b_{ij}. Then, the determinant of $\mathbf{B}$ is given by

$$|\mathbf{B}| \equiv \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \epsilon_{ijk}b_{1i}b_{2j}b_{3k} = \epsilon_{ijk}b_{i1}b_{j2}b_{k3}. \tag{1.4.55}$$

The two expressions in terms of ϵ_{ijk} correspond to the expansion of the determinant by rows and columns, respectively. It is straightforward to verify that (1.4.55) is correct. In fact, those familiar with determinants will recognize that (1.4.55) is the definition of a determinant.

(7) Vector product of $\mathbf{u}$ and $\mathbf{v}$. The ith component is given by

$$(\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk}u_j v_k. \tag{1.4.56}$$

(8) *Curl of a vector*. The ith component is given by

$$(\text{curl } \mathbf{u})_i = (\nabla \times \mathbf{u})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}u_k = \epsilon_{ijk}u_{k,j}. \tag{1.4.57}$$