CAMBRIDGE STUDIES IN
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Completely Bounded Maps and Operator Algebras

In this book the reader is provided with a tour of the principal results and ideas in the theories of completely positive maps, completely bounded maps, dilation theory, operator spaces, and operator algebras, together with some of their main applications.

The author assumes only that the reader has a basic background in functional analysis and C^*-algebras, and the presentation is self-contained and paced appropriately for graduate students new to the subject. The book could be used as a text for a course or for independent reading; with this in mind, many exercises are included. Experts will also want this book for their library, since the author presents new and simpler proofs of some of the major results in the area, and many applications are also included.

This will be an indispensable introduction to the theory of operator spaces for all who want to know more.

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To John, Ival, Effie, Susan, Stephen and Lisa.
My past, present and future.

Contents

Preface

This book is intended to give the reader an introduction to the principal results and ideas in the theories of completely positive maps, completely bounded maps, dilation theory, operator spaces, and operator algebras, together with some of their main applications. It is intended to be self-contained and accessible to any reader who has had a first course in functional analysis that included an introduction to C^*-algebras. It could be used as a text for a course or for independent reading. With this in mind, we have included plenty of exercises.

We have made no attempt at giving a full state-of-the-art exposition of any of these fields. Instead, we have tried to give the reader an introduction to many of the important techniques and results of these fields, together with a feel for their connections and some of the important applications of the ideas. However, we present new proofs and approaches to some of the well-known results in this area, which should make this book of interest to the expert in this area as well as to the beginner.

The quickest route to a result is often not the most illuminating. Consequently, we occasionally present more than one proof of some results. For example, scattered throughout the text and exercises are five different proofs of a key inequality of von Neumann. We feel that such redundancy can lead to a deeper understanding of the material.

In an effort to establish a common core of knowledge that we can assume the reader is familiar with, we have adopted R.G. Douglas's *Banach Algebra Techniques in Operator Theory* as a basic text. Results that appear in that text we have assumed are known, and we have attempted to give a full accounting of all other facts by either presenting them, leaving them as an exercise, or giving a reference. Consequently, parts of the text may seem unnecessarily elementary to some readers. For example, readers with a background in Banach spaces or C^*-algebras may find our discussions of the tensor theory a bit naïve.

We now turn our attention to a description of the contents of this book.

The first seven chapters develop the theory of positive and completely positive maps together with their connections with dilation theory. Dilations are a technique for studying operators on a Hilbert space by representing a given operator as the restriction of a (hopefully) better-understood operator, acting on a larger Hilbert space, to the original space. The operator on the larger space is referred to as a *dilation* of the original operator. Thus, dilation theory involves essentially geometric constructions. We shall see that many of the classic theorems that characterize which sequences of complex numbers are the moments of a measure are really dilation theorems.

One of the better-known dilation theorems is due to Sz.-Nagy and asserts that every contraction operator can be dilated to a unitary operator. Thus to prove some results about contraction operators it is enough to show that they are true for unitary operators. The most famous application of this idea is Sz.-Nagy's elegant proof of an inequality of von Neumann to the effect that the norm of a polynomial in a contraction operator is at most the supremum of the absolute value of the polynomial over the unit disk.

Ando generalized Sz.-Nagy's and von Neumann's results to pairs of commuting contractions, but various analogues of these theorems are known to fail for three or more commuting contractions. Work of Sarason and of Sz.-Nagy and Foias showed that many classical results about analytic functions, including the Nevanlinna–Pick theory, Nehari's theorem, and Caratheodory's completion theorem are consequences of these results about contraction operators. Thus, one finds that there is an operator-theoretic obstruction to generalizing many of these classic results. These results are the focus of Chapter 5.

W.F. Stinespring introduced the theory of completely positive maps as a means of giving abstract necessary and sufficient conditions for the existence of dilations. In many ways completely positive maps play the same role as positive measures when commutative C^*-algebras are replaced by noncommutative C^*-algebras. The connections between completely positive maps and dilation theory were broadened further by Arveson, who developed a deep structure theory for these maps, including an operator-valued Hahn–Banach extension theorem.

Completely positive maps also play a central role in the theory of tensor products of C^*-algebras. Characterizations of nuclear C^*-algebras and injectivity are given in terms of these maps. In noncommutative harmonic analysis they arise in the guise of positive definite operator-valued functions on groups.

In spite of the broad range of applications of completely positive maps, this text is one of the few places where one can find a full introduction to their theory.

In the early 1980s, motivated largely by the work of Wittstock and Haagerup, researchers began extending much of the theory of completely positive maps to the family of completely bounded maps. To the extent that completely positive maps are the analogue of positive measures, completely bounded maps are the analogue of bounded measures.

This newer family of maps also allows for the development of a theory that ties together many questions about the existence or nonexistence of similarities or what are sometimes referred to as skew dilations. Two famous problems of this type are Kadison's and Halmos's similarity conjectures. The theory of completely bounded maps has had an enormous impact on both of these conjectures. Kadison conjectured that every bounded homomorphism of a C^*-algebra into the algebra of operators on a Hilbert space is similar to a $*$-homomorphism. Halmos conjectured that every polynomially bounded operator is similar to a contraction.

In Chapters 8 and 9, we develop the basic theory of completely bounded maps and their connections with similarity questions.

In Chapter 10 we study polynomially bounded operators and present Pisier's counterexample to the Halmos conjecture. The Kadison conjecture still remains unresolved at the time of this writing, but in Chapter 19 we present Pisier's theory of similarity and factorization degrees, which we believe is the most hopeful route towards a solution of the Kadison conjecture.

Attempts to generalize von Neumann's results and the theory of polynomially bounded operators to domains other than the unit disk led to the concepts of spectral and K-spectral sets. We study the applications of the theory of completely bounded maps to these ideas in Chapter 11.

In Chapter 12 we get our first introduction to tensor theory in order to further develop some of the multivariable analogues of von Neumann's inequality.

In order to discuss completely positive or completely bounded maps between two spaces, the domains and ranges of these maps need to be what is known as an *operator system* or *operator space*, respectively. Such spaces arise naturally as subspaces of the space of bounded operators on a Hilbert space, and this is how operator systems and operator spaces were originally defined. However, results of Choi and Effros and of Ruan gave abstract characterizations of operator systems and operator spaces that enabled researchers to treat their theory and the corresponding theories of completely positive and completely bounded maps in a way that was free of dependence on this underlying Hilbert space.

These abstract characterizations have had an impact on this field similar to the impact that the Gelfand–Naimark–Segal theorem has had on the study of C^*-algebras. These characterizations have also allowed for the development of many parallels with ideas from the theory of Banach spaces and bounded linear maps, which have in turn led to a deeper understanding of many results in the theory of C^*-algebras and von Neumann algebras.

We present these characterizations in Chapter 13 and give a quick introduction to the rapidly growing field of operator spaces in Chapter 14, where we carefully examine some of the more important examples of operator spaces.

The abstract characterization of operator spaces led to the Blecher–Ruan–Sinclair abstract characterization of operator algebras. The last chapters of this book are devoted to a development of this theory and some of its applications.

We give two separate developments of the Blecher–Ruan–Sinclair theory. First, we present a new proof based on Hamana's theory of injective envelopes, which we develop in Chapters 15 and 16. We then develop the theory of the Haagerup tensor product and the representation theorems for multilinear maps that are completely bounded in the sense of Christensen and Sinclair, and give a proof of the Blecher–Ruan–Sinclair theorem based on this theory. Our development of the Haagerup tensor theory also uses the theory of the injective envelope in a novel fashion.

The remaining two chapters of the book develop some applications of the Blecher–Ruan–Sinclair theorem. First, we develop the theory of the universal operator algebra of a unital algebra and its applications, including new proofs of Nevanlinna's factorization theorem for analytic functions on the disk and Agler's generalization of Nevanlinna's theorem to analytic functions on the bidisk. Finally, in the last chapter, we present Pisier's theory of the universal operator algebra of an operator space, his results on similarity and factorization degree, and their applications to Kadison's similarity conjecture.

This book grew out of my earlier lecture notes on completely positive and completely bounded maps [161], that have been out-of-print for over a decade.

I would like to acknowledge all the friends and colleagues who have helped to make this book possible. David Blecher, Ken Davidson, Gilles Pisier, and Roger Smith have been tremendous sources of information and ideas. The reader should be grateful to Aristides Katavolos, whose proofreading of selected chapters led to a further polishing of the entire manuscript. Ron Douglas has been a constant source of support throughout my academic career and provided the original impetus to write this book. The author would also like to thank the Mathematics Department at Rice University where portions of this book were written, Roger Astley of Cambridge University Press for his support and advice, the proofreaders at TechBooks for making many of my thoughts flow smoother without altering their mathematical content, and Robin Campbell who typed nearly the entire manuscript and contributed much to its overall look. Finally, without my family's patience and endurance this project would have not been possible.

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