Chapter 1

Introduction

It is assumed throughout this book that the reader is familiar with operator theory and the basic properties of C^*-algebras (see for example [76] and [8, Chapter 1]). We concentrate primarily on giving a self-contained exposition of the theory of completely positive and completely bounded maps between C^*-algebras and the applications of these maps to the study of operator algebras, similarity questions, and dilation theory. In particular, we assume that the reader is familiar with the material necessary for the Gelfand–Naimark–Segal theorem, which states that every C^*-algebra has a one-to-one, $*$-preserving, norm-preserving representation as a norm-closed, $*$-closed algebra of operators on a Hilbert space.

In this chapter we introduce some of the key concepts that will be studied in this book.

As well as having a norm, a C^*-algebra also has an order structure, induced by the cone of positive elements. Recall that an element of a C^*-algebra is *positive* if and only if it is self-adjoint and its spectrum is contained in the nonnegative reals, or equivalently, if it is of the form a^*a for some element a. Since the property of being positive is preserved by $*$-isomorphism, if a C^*-algebra is represented as an algebra of operators on a Hilbert space, then the positive elements of the C^*-algebra coincide with the positive operators that are contained in the representation of the algebra. An equivalent characterization of positivity for an operator on a Hilbert space is that A is a *positive operator* provided that the inner product $\langle Ax, x\rangle$ is nonnegative for every vector x in the space. We shall write $a \geq 0$ to denote that a is positive.

The positive elements in a C^*-algebra $\mathcal{A}$ are a norm-closed, convex cone in the C^*-algebra, denoted by $\mathcal{A}^+$. If h is a self-adjoint element, then it is easy to see, via the functional calculus, that h is the difference of two positive elements.

Indeed, if we let

$$f^+(x) = \begin{cases} x, & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad f^-(x) = \begin{cases} 0, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

then using the functional calculus we have that $h = f^+(h) - f^-(h)$, with $f^+(h)$ and $f^-(h)$ both positive. In particular, we see that the real linear span of the positive elements is the set of self-adjoint elements, which is also norm-closed.

Using the *Cartesian decomposition* of an arbitrary element a of $\mathcal{A}$, namely, $a = h + ik$ with $h = h^*$, $k = k^*$, we see that

$$a = (p_1 - p_2) + i(p_3 - p_4),$$

with p_i positive, $i = 1, 2, 3, 4$. Thus, the complex linear span of $\mathcal{A}^+$ is $\mathcal{A}$.

In addition to having its own norm and order structure, a C^*-algebra is also equipped with a whole sequence of norms and order structures on a set of C^*-algebras naturally associated with the original algebra, and this additional structure will play a central role in this book.

To see how to obtain this additional structure, let $\mathcal{A}$ be our C^*-algebra, let M_n denote the $n \times n$ complex matrices, and let $M_n(\mathcal{A})$ denote the set of $n \times n$ matrices with entries from $\mathcal{A}$. We'll denote a typical element of $M_n(\mathcal{A})$ by $(a_{i,j})$.

There is a natural way to make $M_n(\mathcal{A})$ into a $*$-algebra. Namely, for $(a_{i,j})$ and $(b_{i,j})$ in $M_n(\mathcal{A})$, set

$$(a_{i,j}) \cdot (b_{i,j}) = \left(\sum_{k=1}^{n} a_{i,k} b_{k,j} \right)$$

and

$$(a_{i,j})^* = (a_{j,i}^*).$$

What is not so obvious is that there is a unique way to introduce a norm such that $M_n(\mathcal{A})$ becomes a C^*-algebra.

To see how this is done, we begin with the most basic of all C^*-algebras, $B(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$.

If we let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$, then there is a natural norm and inner product on $\mathcal{H}^{(n)}$ that makes it into a Hilbert space. Namely,

$$\left\| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 = \|h_1\|^2 + \cdots + \|h_n\|^2$$

and

$$\left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} k_1, \\ \vdots \\ k_n \end{pmatrix} \right\rangle_{\mathcal{H}^{(n)}} = \langle h_1, k_1 \rangle_{\mathcal{H}} + \cdots + \langle h_n, k_n \rangle_{\mathcal{H}},$$

where

$$\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \text{ and } \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$$

are in $\mathcal{H}^{(n)}$. This Hilbert space is also often denoted $\ell_n^2(\mathcal{H})$. We prefer to regard elements of $\mathcal{H}^{(n)}$ as column vectors, for reasons that will become apparent shortly.

There is a natural way to regard an element of $M_n(B(\mathcal{H}))$ as a linear map on $\mathcal{H}^{(n)}$, by using the ordinary rules for matrix products. That is, we set

$$(T_{ij}) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} T_{1j} h_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj} h_j \end{pmatrix},$$

for (T_{ij}) in $M_n(B(\mathcal{H}))$ and $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ in $\mathcal{H}^{(n)}$. It is easily checked (Exercise 1.1) that every element of $M_n(B(\mathcal{H}))$ defines a bounded linear operator on $\mathcal{H}^{(n)}$ and that this correspondence yields a one-to-one $*$-isomorphism of $M_n(B(\mathcal{H}))$ onto $B(\mathcal{H}^{(n)})$ (Exercise 1.2). Thus, the identification $M_n(B(\mathcal{H})) = B(\mathcal{H}^{(n)})$ gives us a norm that makes $M_n(B(\mathcal{H}))$ a C^*-algebra.

Now, given any C^*-algebra $\mathcal{A}$, one way that $M_n(\mathcal{A})$ can be viewed as a C^*-algebra is to first choose a one-to-one $*$-representation of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$ so that $\mathcal{A}$ can be identified as a C^*-subalgebra of $B(\mathcal{H})$. This allows us to identify $M_n(\mathcal{A})$ as a $*$-subalgebra of $M_n(B(\mathcal{H}))$. It is straightforward to verify that the image of $M_n(\mathcal{A})$ under this representation is closed and hence a C^*-algebra.

Thus, by using a one-to-one *-representation of $\mathcal{A}$, we have *a* way to turn $M_n(\mathcal{A})$ into a C^*-algebra. But since the norm is unique on a C^*-algebra, we see that the norm on $M_n(\mathcal{A})$ defined in this fashion is independent of the particular representation of $\mathcal{A}$ that we chose. Since positive elements remain positive under $*$-isomorphisms, we see that the positive elements of $M_n(\mathcal{A})$ are also uniquely determined.

So we see that in addition to having a norm and an order, every C^*-algebra $\mathcal{A}$ carries along this extra "baggage" of canonically defined norms and orders on each $M_n(\mathcal{A})$. Remarkably, keeping track of how this extra structure behaves yields far more information than one might expect. The study of these *matrix norms* and *matrix orders* will be a central topic of this book.

For some examples of this structure, we first consider M_k. We can regard this as a C^*-algebra by identifying M_k with the linear transformations on k-dimensional (complex) Hilbert space, $\mathbb{C}^k$. There is a natural way to identify $M_n(M_k)$ with M_{nk}, namely, forget the additional parentheses. It is easy to see that, with this identification, the multiplication and $*$-operation on $M_n(M_k)$ become the usual multiplication and $*$-operation on M_{nk}, that is, the identification defines a $*$-isomorphism. Hence, the unique norm on $M_n(M_k)$ is just the norm obtained by this identification with M_{nk}. An element of $M_n(M_k)$ will be positive if and only if the corresponding matrix in M_{nk} is positive.

For a second example, let X be a compact Hausdorff space, and let $C(X)$ denote the continuous complex-valued functions on X. Setting $f^*(x) = \overline{f(x)}$, we have

$$\|f\| = \sup\{|f(x)|: x \in X\},$$

and defining the algebra operations pointwise makes $C(X)$ into a C^*-algebra. An element $F = (f_{i,j})$ of $M_n(C(X))$ can be thought of as a continuous M_n-valued function. Note that addition, multiplication, and the $*$-operation in $M_n(C(X))$ are just the pointwise addition, pointwise multiplication, and pointwise conjugate-transpose operations of these matrix-valued functions. If we set

$$\|F\| = \sup\{\|F(x)\|: x \in X\},$$

where by $\|F(x)\|$ we mean the norm in M_n, then it is easily seen that this defines a C^*-norm on $M_n(C(X))$, and thus is the unique norm in which $M_n(C(X))$ is a C^*-algebra. Note that the positive elements of $M_n(C(X))$ are those F for which $F(x)$ is a positive matrix for all x.

Now, given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and a map $\phi: \mathcal{A} \to \mathcal{B}$, we also obtain maps $\phi_n: M_n(\mathcal{A}) \to M_n(\mathcal{B})$ via the formula

$$\phi_n((a_{i,j})) = (\phi(a_{i,j})).$$

In general the adverb *completely* means that all of the maps $\{\phi_n\}$ enjoy some property.

For example, the map ϕ is called *positive* if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$, and ϕ is called *completely positive* if every ϕ_n is a positive map.

In a similar fashion, if ϕ is a bounded map, then each ϕ_n will be bounded, and when $\|\phi\|_{\text{cb}} = \sup_n \|\phi_n\|$ is finite, we call ϕ a *completely bounded* map.

One's initial hope is perhaps that C^*-algebras are sufficiently nice that every positive map is completely positive and every bounded map is completely bounded. Indeed, one might expect that $\|\phi\| = \|\phi_n\|$ for all n. For these reasons, we begin with an example of a fairly nice map where those norms are different.

Let $\{E_{i,j}\}_{i,j=1}^2$ denote the system of matrix units for M_2 [that is, $E_{i,j}$ is 1 in the (i, j)th entry and 0 elsewhere], and let ϕ: $M_2 \to M_2$ be the transpose map, so that $\phi(E_{i,j}) = E_{j,i}$. It is easy to verify (Exercise 1.9) that the transpose of a positive matrix is positive and that the norm of the transpose of a matrix is the same as the norm of the matrix, so ϕ is positive and $\|\phi\| = 1$. Now let's consider ϕ_2: $M_2(M_2) \to M_2(M_2)$.

Note that the matrix of matrix units,

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

is positive, but that

$$\phi_2\left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}\right) = \begin{bmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not positive. Thus, ϕ is a positive map but not completely positive. In a similar fashion, we have that

$$\left\| \begin{bmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{bmatrix} \right\| = 1,$$

while the norm of its image under ϕ_2 has norm 2. Thus, $\|\phi_2\| \geq 2$, so $\|\phi_2\| \neq \|\phi\|$. It turns out that ϕ is completely bounded, in fact, $\sup_n \|\phi_n\| = 2$, as we shall see later in this book.

To obtain an example of a map that's not completely bounded, we need to repeat the above example but on an infinite-dimensional space. So let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space with a countable, orthonormal basis, $\{e_n\}_{n=1}^{\infty}$. Every bounded, linear operator T on $\mathcal{H}$ can be thought of as an infinite matrix whose (i, j)th entry is the inner product $\langle Te_j, e_i\rangle$. One then defines a map ϕ from the C^*-algebra of bounded linear operators on $\mathcal{H}$, $B(\mathcal{H})$, to $B(\mathcal{H})$ by the transpose. Again ϕ will be positive and an isometry (Exercise 1.9), but $\|\phi_n\| \geq n$. To see this last claim, let $\{E_{i,j}\}_{i,j=1}^{\infty}$ be matrix units on $\mathcal{H}$, and for fixed n, let $A = (E_{j,i})$, that is, A is the element of $M_n(B(\mathcal{H}))$ whose (i, j)th entry is $E_{j,i}$. We leave it to the reader to verify that $\|A\| = 1$ (in fact, A is a partial isometry), but $\|\phi_n(A)\| = n$ (Exercise 1.8).

There is an alternative approach to the above constructions, via tensor products. A reader familiar with tensor products has perhaps realized that the algebra $M_n(\mathcal{A})$ that we've defined is readily identified with the tensor product algebra $M_n \otimes \mathcal{A}$. Recall that one makes the tensor product of two algebras into an algebra by defining $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$ and then extending linearly. If $\{E_{i,j}\}_{i,j=1}^n$ denotes the canonical basis for M_n, then an element $(a_{i,j})$ in $M_n(\mathcal{A})$ can be identified with $\sum_{i,j=1}^n a_{i,j} \otimes E_{i,j}$ in $M_n \otimes \mathcal{A}$. We leave it to the reader to verify (Exercise 1.10) that with this identification of $M_n(\mathcal{A})$ and $M_n \otimes \mathcal{A}$, the multiplication defined on $M_n(\mathcal{A})$ becomes the tensor product multiplication on $M_n \otimes \mathcal{A}$. Thus, this identification is an isomorphism of these algebras.

We shall on occasion return to this tensor product notation to simplify concepts.

Now that the reader has been introduced to the concepts of completely positive and completely bounded maps, we turn to the topic of dilations.

In general, the key idea behind a dilation is to realize an operator or a mapping into a space of operators as "part" of something simpler on a larger space.

The simplest case is the *unitary dilation of an isometry*. Let V be an isometry on $\mathcal{H}$, and let $P = I_{\mathcal{H}} - VV^*$ be the projection onto the orthocomplement of the range of V. If we define U on $\mathcal{H} \oplus \mathcal{H} = \mathcal{K}$ via

$$U = \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix},$$

then it is easily checked that $U^*U = UU^* = I_{\mathcal{K}}$, so that U is a unitary on $\mathcal{K}$. Moreover, if we identify $\mathcal{H}$ with $\mathcal{H} \oplus 0$, then

$$V^n = P_{\mathcal{H}} U^n|_{\mathcal{H}} \quad \text{for all} \quad n \geq 0.$$

Thus, any isometry V can be realized as the restriction of some unitary to one of its subspaces in a manner that also respects the powers of both operators.

In a similar fashion, one can construct an *isometric dilation of a contraction*. Let T be an operator on $\mathcal{H}$, $\|T\| \leq 1$, and let $D_T = (I - T^*T)^{1/2}$. Note that $\|Th\|^2 + \|D_T h\|^2 = \langle T^*Th, h\rangle + \langle D_T^2 h, h\rangle = \|h\|^2$.

We set

$$\ell^2(\mathcal{H}) = \left\{ (h_1, h_2, \ldots) \colon h_n \in \mathcal{H} \text{ for all } n, \ \sum_{n=1}^{\infty} \|h_n\|^2 < +\infty \right\}.$$

This is a Hilbert space with $\|(h_1, h_2, \ldots)\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2$, and inner product $\langle (h_1, h_2, \ldots), (k_1, k_2, \ldots) \rangle = \sum_{n=1}^{\infty} \langle h_n, k_n \rangle$.

We define $V: \ell^2(\mathcal{H}) \to \ell^2(\mathcal{H})$ via $V((h_1, h_2, \ldots)) = (Th_1, D_T h_1, h_2, \ldots)$. Since $\|V((h_1, h_2, \ldots))\|^2 = \|Th_1\|^2 + \|D_T h_1\|^2 + \|h_2\|^2 + \cdots = \|(h_1, h_2, \ldots)\|^2$, V is an isometry on $\ell^2(\mathcal{H})$. If we identify $\mathcal{H}$ with $\mathcal{H} \oplus 0 \oplus \cdots$, then it is clear that $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$ for all $n \geq 0$.

Combining these two constructions yields the unitary dilation of a contraction.

Theorem 1.1 (Sz.-Nagy's dilation theorem). *Let T be a contraction operator on a Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary operator U on $\mathcal{K}$ such that*

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}.$$

Proof. Let $\mathcal{K} = \ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H})$, and identify $\mathcal{H}$ with $(\mathcal{H} \oplus 0 \oplus \cdots) \oplus 0$. Let V be the isometric dilation of T on $\ell^2(\mathcal{H})$, and let U be the unitary dilation of V on $\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H})$. Since $\mathcal{H} \subseteq \ell^2(\mathcal{H}) \oplus 0$, we have that $P_{\mathcal{H}} U^n|_{\mathcal{H}} = P_{\mathcal{H}} V^n|_{\mathcal{H}} = T^n$ for all $n \geq 0$. □

Whenever Y is an operator on a Hilbert space $\mathcal{K}$, $\mathcal{H}$ is a subspace of $\mathcal{K}$, and $X = P_{\mathcal{H}} Y|_{\mathcal{H}}$, then we call X a *compression* of Y.

There is a certain sense in which a "minimal" unitary dilation can be chosen, and this dilation is in some sense unique. We shall not need these facts now, but shall return to them in Chapter 4.

To see the power of this simple geometric construction, we now give Sz.-Nagy's proof of an inequality due to von Neumann.

Corollary 1.2 (von Neumann's inequality). *Let T be a contraction on a Hilbert space. Then for any polynomial p,*

$$\|p(T)\| \leq \sup\{|p(z)|: |z| \leq 1\}.$$

Proof. Let U be a unitary dilation of T. Since $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$ for all $n \geq 0$, it follows, by taking linear combinations, that $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$, and hence $\|p(T)\| \leq \|p(U)\|$. Since unitaries are normal operators, we have that $\|p(U)\| = \sup\{|p(\lambda)|: \lambda \in \sigma(U)\}$, where $\sigma(U)$ denotes the spectrum of U. Finally, since U is unitary, $\sigma(U)$ is contained in the unit circle and the result follows. □

In Chapter 2, we will give another proof of von Neumann's inequality, using some facts about positive maps, and then in Chapter 4 we will obtain Sz.-Nagy's dilation theorem as a consequence of von Neumann's inequality.

Exercises

1.1 Let (T_{ij}) be in $M_n(B(\mathcal{H}))$. Verify that the linear transformation it defines on $\mathcal{H}^{(n)}$ is bounded and that, in fact, $\|(T_{ij})\| \le (\sum_{i,j=1}^n \|T_{ij}\|^2)^{1/2}$.

1.2 Let $\pi\colon M_n(B(\mathcal{H})) \to B(\mathcal{H}^{(n)})$ be the identification given in the text.

(i) Verify that π is a one-to-one $*$-homomorphism.

(ii) Let $E_j\colon \mathcal{H} \to \mathcal{H}^{(n)}$ be the map defined by setting $E_j(h)$ equal to the vector that has h for its jth component and is 0 elsewhere. Show that $E_j^*\colon \mathcal{H}^{(n)} \to \mathcal{H}$ is the map that sends a vector in $\mathcal{H}^{(n)}$ to its jth component.

(iii) Given $T \in B(\mathcal{H}^{(n)})$, set $T_{ij} = E_i^* T E_j$. Show that $\pi((T_{ij})) = T$ and that consequently π is onto.

1.3 Let (T_{ij}) be in $M_n(B(\mathcal{H}))$. Prove that (T_{ij}) is a contraction if and only if for every choice of $2n$ unit vectors $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathcal{H}$, the scalar matrix $(\langle T_{ij}x_j, y_i\rangle)$ is a contraction.

1.4 Let (T_{ij}) be in $M_n(B(\mathcal{H}))$. Prove that (T_{ij}) is positive if and only if for every choice of n vectors $x_1, \ldots, x_n$ in $\mathcal{H}$ the scalar matrix $(\langle T_{ij}x_j, x_i\rangle)$ is positive.

1.5 Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, and let $\pi\colon \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism with $\pi(1) = 1$. Show that π is completely positive and completely bounded and that $\|\pi\| = \|\pi_n\| = \|\pi\|_{\text{cb}} = 1$.

1.6 Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be C^*-algebras, and let $\phi\colon \mathcal{A} \to \mathcal{B}$ and $\psi\colon \mathcal{B} \to \mathcal{C}$ be (completely) positive maps. Show that $\psi \circ \phi$ is (completely) positive.

1.7 Let $\{E_{i,j}\}_{i,j=1}^n$ be matrix units for M_n, let $A = (E_{j,i})_{i,j=1}^n$, and let $B = (E_{i,j})_{i,j=1}^n$ be in $M_n(M_n)$. Show that A is unitary and that $\frac{1}{n}B$ is a rank one projection.

1.8 Let $\{E_{i,j}\}_{i,j=1}^\infty$ be a system of matrix units for $B(\mathcal{H})$, let $A = (E_{j,i})_{i,j=1}^n$, and let $B = (E_{i,j})_{i,j=1}^n$ be in $M_n(B(\mathcal{H}))$. Show that A is a partial isometry, and that $\frac{1}{n}B$ is a projection. Show that $\phi_n(A) = B$ and $\|\phi_n(A)\| = n$.

1.9 Let A be in M_n, and let A^t denote the transpose of A. Prove that A is positive if and only if A^t is positive, and that $\|A\| = \|A^t\|$. Prove that these same results hold for operators on a separable, infinite-dimensional Hilbert space, when we fix an orthonormal basis, regard operators as infinite matrices, and use this to define a transpose map.

1.10 Prove that the map $\pi\colon M_n(\mathcal{A}) \to M_n \otimes \mathcal{A}$ defined by $\pi((a_{i,j})) = \sum_{i,j=1}^n a_{i,j} \otimes E_{i,j}$ is an algebra isomorphism.

Chapter 2

Positive Maps

Before turning our attention to the completely positive or completely bounded maps, we begin with some results on positive maps that we shall need repeatedly. These results also serve to illustrate how many simplifications arise when one passes to the smaller class of completely positive maps.

If $\mathcal{S}$ is a subset of a C^*-algebra $\mathcal{A}$, then we set

$$S^* = \{a : a^* \in \mathcal{S}\},$$

and we call $\mathcal{S}$ *self-adjoint* when $\mathcal{S} = \mathcal{S}^*$. If $\mathcal{A}$ has a unit 1 and $\mathcal{S}$ is a self-adjoint subspace of $\mathcal{A}$ containing 1, then we call $\mathcal{S}$ an *operator system*. If $\mathcal{S}$ is an operator system and h is a self-adjoint element of S, then even though $f^+(h)$ and $f^-(h)$ need not belong to $\mathcal{S}$ (since these only belong to the norm-closed algebra generated by h), we can still write h as the difference of two positive elements in $\mathcal{S}$. Indeed,

$$h = \frac{1}{2}(\|h\| \cdot 1 + h) - \frac{1}{2}(\|h\| \cdot 1 - h).$$

If $\mathcal{S}$ is an operator system, $\mathcal{B}$ is a C^*-algebra, and $\phi \colon \mathcal{S} \to \mathcal{B}$ is a linear map, then ϕ is called a *positive map* provided that it maps positive elements of $\mathcal{S}$ to positive elements of $\mathcal{B}$. In this chapter, we develop some of the properties of positive maps. In particular, we shall be concerned with how the assumption of positivity is related to the norm of the map, and conversely, when assumptions about the norm of a map guarantee that it is positive. We give a fairly elementary proof of von Neumann's inequality (Corollary 2.7), which only uses these observations about positive maps and an elementary result from complex analysis due to Fejer and Riesz.

If ϕ is a positive, linear functional on an operator system $\mathcal{S}$, then it is easy to show that $\|\phi\| = \phi(1)$ (Exercise 2.3). When the range is a C^*-algebra the situation is quite different.

Proposition 2.1. *Let $\mathcal{S}$ be an operator system, and let $\mathcal{B}$ be a C^*-algebra. If $\phi\colon \mathcal{S} \to \mathcal{B}$ is a positive map, then ϕ is bounded and*

$$\|\phi\| \le 2\|\phi(1)\|.$$

Proof. First note that if p is positive, then $0 \le p \le \|p\| \cdot 1$ and so $0 \le \phi(p) \le \|p\| \cdot \phi(1)$, from which it follows that $\|\phi(p)\| \le \|p\| \cdot \|\phi(1)\|$ when $p \ge 0$.

Next note that if p_1 and p_2 are positive, then $\|p_1 - p_2\| \le \max\{\|p_1\|, \|p_2\|\}$. If h is self-adjoint in $\mathcal{S}$, then using the above decomposition of h, we have

$$\phi(h) = \frac{1}{2}\phi(\|h\| \cdot 1 + h) - \frac{1}{2}\phi(\|h\| \cdot 1 - h),$$

which expresses $\phi(h)$ as a difference of two positive elements of $\mathcal{B}$. Thus,

$$\|\phi(h)\| \le \frac{1}{2}\max\{\|\phi(\|h\| \cdot 1 + h)\|, \|\phi(\|h\| \cdot 1 - h)\|\} \le \|h\| \cdot \|\phi(1)\|.$$

Finally, if a is an arbitrary element of $\mathcal{S}$, then $a = h + ik$ with $\|h\|, \|k\| \le \|a\|$, $h = h^*$, $k = k^*$, and so

$$\|\phi(a)\| \le \|\phi(h)\| + \|\phi(k)\| \le 2\|a\| \cdot \|\phi(1)\|. \qquad \square$$

Let us reproduce an example of Arveson, which shows that 2 is the best constant in Proposition 2.1.

Example 2.2. Let $\mathbb{T}$ denote the unit circle in the complex plane, $C(\mathbb{T})$ the continuous functions on $\mathbb{T}$, z the coordinate function, and $\mathcal{S} \subseteq C(\mathbb{T})$ the subspace spanned by 1, z, and $\bar{z}$.

We define $\phi\colon \mathcal{S} \to M_2$ by

$$\phi(a + bz + c\bar{z}) = \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}.$$

We leave it to the reader to verify that an element $a1 + bz + c\bar{z}$ of $\mathcal{S}$ is positive if and only if $c = \bar{b}$ and $a \ge 2|b|$. It is fairly standard that a self-adjoint element of M_2 is positive if and only if its diagonal entries and its determinant are nonnegative real numbers. Combining these two facts, it is clear that ϕ is a positive map. However,

$$2\|\phi(1)\| = 2 = \|\phi(z)\| \le \|\phi\|,$$

so that $\|\phi\| = 2\|\phi(1)\|$.

The existence of unital, positive maps that are not contractive can be roughly attributed to two factors. One is the noncommutativity of the range, the other