Differential Equations

Linear, Nonlinear, Ordinary, Partial

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Many physical, chemical and biological systems can be described using mathematical models. Once the model is formulated, we usually need to solve a differential equation in order to predict and quantify the features of the system being modelled. As a precursor to this, we consider linear, second order ordinary differential equations of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = F(x),$$

with $P(x)$, $Q(x)$ and $R(x)$ finite polynomials that contain no common factor. This equation is inhomogeneous and has variable coefficients. The form of these polynomials varies according to the underlying physical problem that we are studying. However, we will postpone any discussion of the physical origin of such equations until we have considered some classical mathematical models in Chapters 2 and 3.

After dividing through by $P(x)$, we obtain the more convenient, equivalent form,

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x). \tag{1.1}$$

This process is mathematically legitimate, provided that $P(x) \neq 0$. If $P(x_0) = 0$ at some point $x = x_0$, it is *not* legitimate, and we call x_0 a **singular** point of the equation. If $P(x_0) \neq 0$, x_0 is a **regular** or **ordinary** point of the equation. If $P(x) \neq 0$ for all points x in the interval where we want to solve the equation, we say that the equation is **nonsingular**, or **regular**, on the interval.

We usually need to solve (1.1) subject to either **initial conditions** of the form $y(a) = \alpha$, $y'(a) = \beta$ or **boundary conditions**, of which $y(a) = \alpha$ and $y(b) = \beta$ are typical examples. It is worth reminding ourselves that, given the ordinary differential equation and initial conditions (an **initial value problem**), the objective is to determine the solution for other values of x, typically, $x > a$, as illustrated in Figure 1.1. As an example, consider a projectile. The initial conditions are the position of the projectile and the speed and angle to the horizontal at which it is fired. We then want to know the path of the projectile, given these initial conditions.

For initial value problems of this form, it is possible to show that:

(i) If $a_1(x)$, $a_0(x)$ and $f(x)$ are continuous on some open interval I that contains the initial point a, a unique solution of the initial value problem exists on the interval I, as we shall demonstrate in Chapter 8.

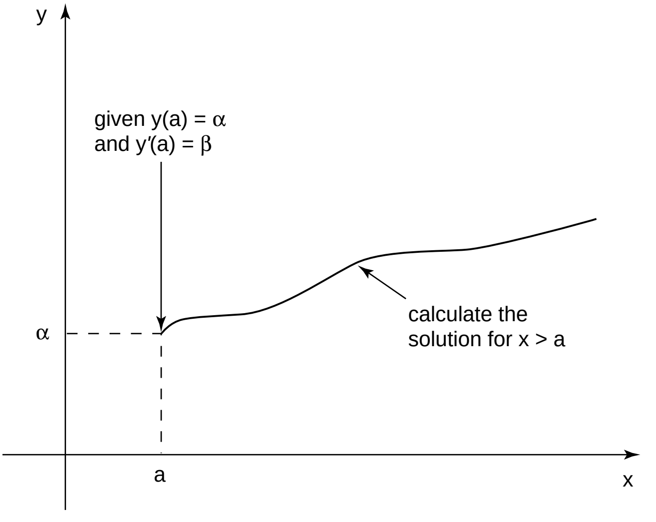

Fig. 1.1. An initial value problem.

(ii) The structure of the solution of the initial value problem is of the form

$$y = \underbrace{A\,u_1(x) + B\,u_2(x)}_{\text{Complementary function}} + \underbrace{G(x)}_{\text{Particular integral}},$$

where A, B are constants that are fixed by the initial conditions and $u_1(x)$ and $u_2(x)$ are linearly independent solutions of the corresponding homogeneous problem $y'' + a_1(x)y' + a_0(x)y = 0$.

These results can be proved rigorously, but nonconstructively, by studying the operator

$$Ly \equiv \frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y,$$

and regarding $L : C^2(I) \to C^0(I)$ as a linear transformation from the space of twice-differentiable functions defined on the interval I to the space of continuous functions defined on I. The solutions of the homogeneous equation are elements of the null space of L. This subspace is completely determined once its basis is known. The solution of the inhomogeneous problem, $Ly = f$, is then given formally as $y = L^{-1}f$. Unfortunately, if we actually want to construct the solution of a particular equation, there is a lot more work to do.

Before we try to construct the general solution of the inhomogeneous initial value problem, we will outline a series of subproblems that are more tractable.

1.1 The Method of Reduction of Order

As a first simplification we discuss the solution of the homogeneous differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \tag{1.2}$$

on the assumption that we know one solution, say $y(x) = u_1(x)$, and only need to find the second solution. We will look for a solution of the form $y(x) = U(x)u_1(x)$. Differentiating $y(x)$ using the product rule gives

$$\frac{dy}{dx} = \frac{dU}{dx}u_1 + U\frac{du_1}{dx},$$

$$\frac{d^2y}{dx^2} = \frac{d^2U}{dx^2}u_1 + 2\frac{dU}{dx}\frac{du_1}{dx} + U\frac{d^2u_1}{dx^2}.$$

If we substitute these expressions into (1.2) we obtain

$$\frac{d^2U}{dx^2}u_1 + 2\frac{dU}{dx}\frac{du_1}{dx} + U\frac{d^2u_1}{dx^2} + a_1(x)\left(\frac{dU}{dx}u_1 + U\frac{du_1}{dx}\right) + a_0(x)Uu_1 = 0.$$

We can now collect terms to get

$$U\left(\frac{d^2u_1}{dx^2} + a_1(x)\frac{du_1}{dx} + a_0(x)u_1\right) + u_1\frac{d^2U}{dx^2} + \frac{dU}{dx}\left(2\frac{du_1}{dx} + a_1u_1\right) = 0.$$

Now, since $u_1(x)$ is a solution of (1.2), the term multiplying U is zero. We have therefore obtained a differential equation for dU/dx, and, by defining $Z = dU/dx$, have

$$u_1\frac{dZ}{dx} + Z\left(2\frac{du_1}{dx} + a_1u_1\right) = 0.$$

Dividing through by Zu_1 we have

$$\frac{1}{Z}\frac{dZ}{dx} + \frac{2}{u_1}\frac{du_1}{dx} + a_1 = 0,$$

which can be integrated directly to yield

$$\log|Z| + 2\log|u_1| + \int^x a_1(s)\,ds = C,$$

where s is a dummy variable, for some constant C. Thus

$$Z = \frac{c}{u_1^2}\exp\left\{-\int^x a_1(s)\,ds\right\} = \frac{dU}{dx}$$

where $c = e^C$. This can then be integrated to give

$$U(x) = \int^x \frac{c}{u_1^2(t)}\exp\left\{-\int^t a_1(s)\,ds\right\}dt + \tilde{c},$$

for some constant $\tilde{c}$. The solution is therefore

$$y(x) = u_1(x)\int^x \frac{c}{u_1^2(t)} \exp\left\{-\int^t a_1(s)\,ds\right\}\,dt + \tilde{c}u_1(x).$$

We can recognize $\tilde{c}u_1(x)$ as the part of the complementary function that we knew to start with, and

$$u_2(x) = u_1(x)\int^x \frac{1}{u_1^2(t)} \exp\left\{-\int^t a_1(s)ds\right\}dt \tag{1.3}$$

as the second part of the complementary function. This result is called **the reduction of order formula**.

Example

Let's try to determine the full solution of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0,$$

given that $y = u_1(x) = x$ is a solution. We firstly write the equation in standard form as

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{2}{1-x^2}y = 0.$$

Comparing this with (1.2), we have $a_1(x) = -2x/(1-x^2)$. After noting that

$$\int^t a_1(s)\,ds = \int^t -\frac{2s}{1-s^2}\,ds = \log(1-t^2),$$

the reduction of order formula gives

$$u_2(x) = x\int^x \frac{1}{t^2}\exp\left\{-\log(1-t^2)\right\}\,dt = x\int^x \frac{dt}{t^2(1-t^2)}.$$

We can express the integrand in terms of its partial fractions as

$$\frac{1}{t^2(1-t^2)} = \frac{1}{t^2} + \frac{1}{1-t^2} = \frac{1}{t^2} + \frac{1}{2(1+t)} + \frac{1}{2(1-t)}.$$

This gives the second solution of (1.2) as

$$u_2(x) = x\int^x \left\{\frac{1}{t^2} + \frac{1}{2(1+t)} + \frac{1}{2(1-t)}\right\}\,dt$$

$$= x\left[-\frac{1}{t} + \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)\right]^x = \frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1,$$

and hence the general solution is

$$y = Ax + B\left\{\frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1\right\}.$$

1.2 The Method of Variation of Parameters

Let's now consider how to find the particular integral *given* the complementary function, comprising $u_1(x)$ and $u_2(x)$. As the name of this technique suggests, we take the constants in the complementary function to be variable, and assume that

$$y = c_1(x)u_1(x) + c_2(x)u_2(x).$$

Differentiating, we find that

$$\frac{dy}{dx} = c_1\frac{du_1}{dx} + u_1\frac{dc_1}{dx} + c_2\frac{du_2}{dx} + u_2\frac{dc_2}{dx}.$$

We will choose to impose the condition

$$u_1\frac{dc_1}{dx} + u_2\frac{dc_2}{dx} = 0, \tag{1.4}$$

and thus have

$$\frac{dy}{dx} = c_1\frac{du_1}{dx} + c_2\frac{du_2}{dx},$$

which, when differentiated again, yields

$$\frac{d^2y}{dx^2} = c_1\frac{d^2u_1}{dx^2} + \frac{du_1}{dx}\frac{dc_1}{dx} + c_2\frac{d^2u_2}{dx^2} + \frac{du_2}{dx}\frac{dc_2}{dx}.$$

This form can then be substituted into the original differential equation to give

$$c_1\frac{d^2u_1}{dx^2} + \frac{du_1}{dx}\frac{dc_1}{dx} + c_2\frac{d^2u_2}{dx^2} + \frac{du_2}{dx}\frac{dc_2}{dx} + a_1\left(c_1\frac{du_1}{dx} + c_2\frac{du_2}{dx}\right) + a_0\left(c_1u_1 + c_2u_2\right) = f.$$

This can be rearranged to show that

$$c_1\left(\frac{d^2u_1}{dx^2} + a_1\frac{du_1}{dx} + a_0u_1\right) + c_2\left(\frac{d^2u_2}{dx^2} + a_1\frac{du_2}{dx} + a_0u_2\right) + \frac{du_1}{dx}\frac{dc_1}{dx} + \frac{du_2}{dx}\frac{dc_2}{dx} = f.$$

Since u_1 and u_2 are solutions of the homogeneous equation, the first two terms are zero, which gives us

$$\frac{du_1}{dx}\frac{dc_1}{dx} + \frac{du_2}{dx}\frac{dc_2}{dx} = f. \tag{1.5}$$

We now have two simultaneous equations, (1.4) and (1.5), for $c_1' = dc_1/dx$ and $c_2' = dc_2/dx$, which can be written in matrix form as

$$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

These can easily be solved to give

$$c_1' = -\frac{fu_2}{W}, \quad c_2' = \frac{fu_1}{W},$$

where

$$W = u_1u_2' - u_2u_1' = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

is called the **Wronskian**. These expressions can be integrated to give

$$c_1 = \int^x -\frac{f(s)u_2(s)}{W(s)}\,ds + A, \quad c_2 = \int^x \frac{f(s)u_1(s)}{W(s)}\,ds + B.$$

We can now write down the solution of the entire problem as

$$y(x) = u_1(x)\int^x -\frac{f(s)u_2(s)}{W(s)}\,ds + u_2(x)\int^x \frac{f(s)u_1(s)}{W(s)}\,ds + Au_1(x) + Bu_2(x).$$

The particular integral is therefore

$$y(x) = \int^x f(s)\left\{\frac{u_1(s)u_2(x) - u_1(x)u_2(s)}{W(s)}\right\} ds. \tag{1.6}$$

This is called the **variation of parameters formula**.

Example

Consider the equation

$$\frac{d^2y}{dx^2} + y = x\sin x.$$

The homogeneous form of this equation has constant coefficients, with solutions

$$u_1(x) = \cos x, \quad u_2(x) = \sin x.$$

The variation of parameters formula then gives the particular integral as

$$y = \int^x s\sin s\left\{\frac{\cos s\sin x - \cos x\sin s}{1}\right\} ds,$$

since

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

We can split the particular integral into two integrals as

$$y(x) = \sin x\int^x s\sin s\cos s\,ds - \cos x\int^x s\sin^2 s\,ds$$

$$= \frac{1}{2}\sin x\int^x s\sin 2s\,ds - \frac{1}{2}\cos x\int^x s\left(1 - \cos 2s\right)\,ds.$$

Using integration by parts, we can evaluate this, and find that

$$y(x) = -\frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x + \frac{1}{8}\cos x$$

is the required particular integral. The general solution is therefore

$$y = c_1\cos x + c_2\sin x - \frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x.$$

Although we have given a rational derivation of the reduction of order and variation of parameters formulae, we have made no comment so far about why the procedures we used in the derivation should work at all! It turns out that this has a close connection with the theory of continuous groups, which we will investigate in Chapter 10.

1.2.1 The Wronskian

Before we carry on, let's pause to discuss some further properties of the Wronskian. Recall that if V is a vector space over $\mathbb{R}$, then two elements $\mathbf{v}_1, \mathbf{v}_2 \in V$ are linearly dependent if $\exists\, \alpha_1, \alpha_2 \in \mathbb{R}$, with α_1 and α_2 not both zero, such that $\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2 = \mathbf{0}$.

Now let $V = C^1(a, b)$ be the set of once-differentiable functions over the interval $a < x < b$. If $u_1, u_2 \in C^1(a, b)$ are linearly dependent, $\exists\ \alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 u_1(x) + \alpha_2 u_2(x) = 0\ \forall x \in (a, b)$. Notice that, by direct differentiation, this also gives $\alpha_1 u_1'(x) + \alpha_2 u_2'(x) = 0$ or, in matrix form,

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These are homogeneous equations of the form

$$\mathrm{A}\mathbf{x} = \mathbf{0},$$

which only have nontrivial solutions if $\det(\mathrm{A}) = 0$, that is

$$W = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = u_1 u_2' - u_1' u_2 = 0.$$

In other words, the Wronskian of two linearly dependent functions is identically zero on (a, b). The contrapositive of this result is that if $W \not\equiv 0$ on (a, b), then u_1 and u_2 are linearly independent on (a, b).

Example

The functions $u_1(x) = x^2$ and $u_2(x) = x^3$ are linearly independent on the interval $(-1, 1)$. To see this, note that, since $u_1(x) = x^2$, $u_2(x) = x^3$, $u_1'(x) = 2x$, and $u_2'(x) = 3x^2$, the Wronskian of these two functions is

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4.$$

This quantity is not identically zero, and hence x^2 and x^3 are linearly independent on $(-1, 1)$.

Example

The functions $u_1(x) = f(x)$ and $u_2(x) = kf(x)$, with k a constant, are linearly dependent on any interval, since their Wronskian is

$$W = \begin{vmatrix} f & kf \\ f' & kf' \end{vmatrix} = 0.$$

If the functions u_1 and u_2 are solutions of (1.2), we can show by differentiating $W = u_1 u_2' - u_1' u_2$ directly that

$$\frac{dW}{dx} + a_1(x) W = 0.$$

This first order differential equation has solution

$$W(x) = W(x_0)\ \exp\left\{-\int_{x_0}^{x} a_1(t)dt\right\}, \tag{1.7}$$

which is known as **Abel's formula**. This gives us an easy way of finding the Wronskian of the solutions of any second order differential equation without having to construct the solutions themselves.

Example

Consider the equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{x^2}\right)y = 0.$$

Using Abel's formula, this has Wronskian

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{dt}{t}\right\} = \frac{x_0 W(x_0)}{x} = \frac{A}{x}$$

for some constant A. To find this constant, it is usually necessary to know more about the solutions $u_1(x)$ and $u_2(x)$. We will describe a technique for doing this in Section 1.3.

We end this section with a couple of useful theorems.

Theorem 1.1 *If u_1 and u_2 are linearly independent solutions of the homogeneous, nonsingular ordinary differential equation (1.2), then the Wronskian is either strictly positive or strictly negative.*

Proof From Abel's formula, and since the exponential function does not change sign, the Wronskian is identically positive, identically negative or identically zero. We just need to exclude the possibility that W is ever zero. Suppose that $W(x_1) = 0$. The vectors $\begin{pmatrix} u_1(x_1) \\ u_1'(x_1) \end{pmatrix}$ and $\begin{pmatrix} u_2(x_1) \\ u_2'(x_1) \end{pmatrix}$ are then linearly dependent, and hence $u_1(x_1) = ku_2(x_1)$ and $u_1'(x) = ku_2'(x)$ for some constant k. The function $u(x) = u_1(x) - ku_2(x)$ is also a solution of (1.2) by linearity, and satisfies the initial conditions $u(x_1) = 0$, $u'(x_1) = 0$. Since (1.2) has a unique solution, the obvious solution, $u \equiv 0$, is the only solution. This means that $u_1 \equiv ku_2$. Hence u_1 and u_2 are linearly dependent – a contradiction. □

The nonsingularity of the differential equation is crucial here. If we consider the equation $x^2y'' - 2xy' + 2y = 0$, which has $u_1(x) = x^2$ and $u_2(x) = x$ as its linearly independent solutions, the Wronksian is $-x^2$, which vanishes at $x = 0$. This is because the coefficient of y'' also vanishes at $x = 0$.

Theorem 1.2 (The Sturm separation theorem) *If $u_1(x)$ and $u_2(x)$ are the linearly independent solutions of a nonsingular, homogeneous equation, (1.2), then* *the zeros of* $u_1(x)$ *and* $u_2(x)$ *occur alternately. In other words, successive zeros of* $u_1(x)$ *are separated by successive zeros of* $u_2(x)$ *and vice versa.*

Proof Suppose that x_1 and x_2 are successive zeros of $u_2(x)$, so that $W(x_i) = u_1(x_i)u_2'(x_i)$ for $i = 1$ or 2. We also know that $W(x)$ is of one sign on $[x_1, x_2]$, since $u_1(x)$ and $u_2(x)$ are linearly independent. This means that $u_1(x_i)$ and $u_2'(x_i)$ are nonzero. Now if $u_2'(x_1)$ is positive then $u_2'(x_2)$ is negative (or vice versa), since $u_2(x_2)$ is zero. Since the Wronskian cannot change sign between x_1 and x_2, $u_1(x)$ must change sign, and hence u_1 has a zero in $[x_1, x_2]$, as we claimed. □

As an example of this, consider the equation $y'' + \omega^2 y = 0$, which has solution $y = A\sin\omega x + B\cos\omega x$. If we consider any two of the zeros of $\sin\omega x$, it is immediately clear that $\cos\omega x$ has a zero between them.

1.3 Solution by Power Series: The Method of Frobenius

Up to this point, we have considered ordinary differential equations for which we know at least one solution of the homogeneous problem. From this we have seen that we can easily construct the second independent solution and, in the inhomogeneous case, the particular integral. We now turn our attention to the more difficult case, in which we cannot determine a solution of the homogeneous problem by inspection. We must devise a method that is capable of solving variable coefficient ordinary differential equations in general. As we noted at the start of the chapter, we will restrict our attention to the case where the variable coefficients are simple polynomials. This suggests that we can look for a solution of the form

$$y = x^c \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+c}, \tag{1.8}$$

and hence

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1}, \tag{1.9}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2}, \tag{1.10}$$

where the constants $c, a_0, a_1, \dots$, are as yet undetermined. This is known as the **method of Frobenius**. Later on, we will give some idea of why and when this method can be used. For the moment, we will just try to make it work. We proceed by example, with the simplest case first.

1.3.1 The Roots of the Indicial Equation Differ by an Integer

Consider the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right) y = 0. \tag{1.11}$$

We substitute (1.8) to (1.10) into (1.11), which gives

$$x^2\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)x^{n+c-2}+x\sum_{n=0}^{\infty}a_n(n+c)x^{n+c-1}$$

$$+\left(x^2-\frac{1}{4}\right)\sum_{n=0}^{\infty}a_nx^{n+c}=0.$$

We can rearrange this slightly to obtain

$$\sum_{n=0}^{\infty}a_n\left\{(n+c)(n+c-1)+(n+c)-\frac{1}{4}\right\}x^{n+c}+\sum_{n=0}^{\infty}a_nx^{n+c+2}=0,$$

and hence, after simplifying the terms in the first summation,

$$\sum_{n=0}^{\infty}a_n\left\{(n+c)^2-\frac{1}{4}\right\}x^{n+c}+\sum_{n=0}^{\infty}a_nx^{n+c+2}=0.$$

We now extract the first two terms from the first summation to give

$$a_0\left(c^2-\frac{1}{4}\right)x^c+a_1\left\{(c+1)^2-\frac{1}{4}\right\}x^{c+1}$$

$$+\sum_{n=2}^{\infty}a_n\left\{(n+c)^2-\frac{1}{4}\right\}x^{n+c}+\sum_{n=0}^{\infty}a_nx^{n+c+2}=0. \tag{1.12}$$

Notice that the first term is the only one containing x^c and similarly for the second term in x^{c+1}.

The two summations in (1.12) begin at the same power of x, namely x^{2+c}. If we let $m=n+2$ in the last summation (notice that if $n=0$ then $m=2$, and $n=\infty$ implies that $m=\infty$), (1.12) becomes

$$a_0\left(c^2-\frac{1}{4}\right)x^c+a_1\left\{(c+1)^2-\frac{1}{4}\right\}x^{c+1}$$

$$+\sum_{n=2}^{\infty}a_n\left\{(n+c)^2-\frac{1}{4}\right\}x^{n+c}+\sum_{m=2}^{\infty}a_{m-2}x^{m+c}=0.$$

Since the variables in the summations are merely dummy variables,

$$\sum_{m=2}^{\infty}a_{m-2}x^{m+c}=\sum_{n=2}^{\infty}a_{n-2}x^{n+c},$$

and hence

$$a_0\left(c^2-\frac{1}{4}\right)x^c+a_1\left\{(c+1)^2-\frac{1}{4}\right\}x^{c+1}$$

$$+\sum_{n=2}^{\infty}a_n\left\{(n+c)^2-\frac{1}{4}\right\}x^{n+c}+\sum_{n=2}^{\infty}a_{n-2}x^{n+c}=0.$$

Since the last two summations involve identical powers of x, we can combine them to obtain

$$a_0\left(c^2-\frac{1}{4}\right)x^c+a_1\left\{(c+1)^2-\frac{1}{4}\right\}x^{c+1}$$

$$+\sum_{n=2}^{\infty}\left[a_n\left\{(n+c)^2-\frac{1}{4}\right\}+a_{n-2}\right]x^{n+c}=0. \tag{1.13}$$

Although the operations above are straightforward, we need to take some care to avoid simple slips.

Since (1.13) must hold for *all* values of x, the coefficient of each power of x must be zero. The coefficient of x^c is therefore

$$a_0\left(c^2-\frac{1}{4}\right)=0.$$

Up to this point, most Frobenius analysis is very similar. It is here that the different structures come into play. If we were to use the solution $a_0=0$, the series (1.8) would have a_1x^{c+1} as its first term. This is just equivalent to increasing c by 1. We therefore assume that $a_0\neq 0$, which means that c must satisfy $c^2-\frac{1}{4}=0$. This is called the **indicial equation**, and implies that $c=\pm\frac{1}{2}$. Now, progressing to the next term, proportional to x^{c+1}, we find that

$$a_1\left\{(c+1)^2-\frac{1}{4}\right\}=0.$$

Choosing $c=\frac{1}{2}$ gives $a_1=0$, and, if we were to do this, we would find that we had constructed a solution with one arbitrary constant. However, if we choose $c=-\frac{1}{2}$ the indicial equation is satisfied for arbitrary values of a_1, and a_1 will act as the second arbitrary constant for the solution. In order to generate this more general solution, we therefore let $c=-\frac{1}{2}$.

We now progress to the individual terms in the summation. The general term yields

$$a_n\left\{\left(n-\frac{1}{2}\right)^2-\frac{1}{4}\right\}+a_{n-2}=0 \quad \text{for } n=2,3,\dots .$$

This is called a **recurrence relation**. We solve it by *observation* as follows. We start by rearranging to give

$$a_n=-\frac{a_{n-2}}{n(n-1)}. \tag{1.14}$$

By putting $n=2$ in (1.14) we obtain

$$a_2=-\frac{a_0}{2\cdot 1}.$$

For $n=3$,

$$a_3=-\frac{a_1}{3\cdot 2}.$$

For $n = 4$,

$$a_4 = -\frac{a_2}{4 \cdot 3},$$

and substituting for a_2 in terms of a_0 gives

$$a_4 = -\frac{1}{4 \cdot 3}\left(-\frac{a_0}{2 \cdot 1}\right) = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}.$$

Similarly for $n = 5$, using the expression for a_3 in terms of a_1,

$$a_5 = -\frac{a_3}{5 \cdot 4} = -\frac{1}{5 \cdot 4}\left(-\frac{a_1}{3 \cdot 2}\right) = \frac{a_1}{5!}.$$

A pattern is emerging here and we propose that

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}, \quad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}. \tag{1.15}$$

This can be proved in a straightforward manner by induction, although we will not dwell upon the details here.†

We can now deduce the full solution. Starting from (1.8), we substitute $c = -\frac{1}{2}$, and write out the first few terms in the summation

$$y = x^{-1/2}(a_0 + a_1 x + a_2 x^2 + \cdots).$$

Now, using the forms of the even and odd coefficients given in (1.15),

$$y = x^{-1/2}\left(a_0 + a_1 x - \frac{a_0 x^2}{2!} - \frac{a_1 x^3}{3!} + \frac{a_0 x^4}{4!} + \frac{a_1 x^5}{5!} + \cdots\right).$$

This series splits naturally into two proportional to a_0 and a_1, namely

$$y = x^{-1/2} a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + x^{-1/2} a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

The solution is therefore

$$y(x) = a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}},$$

since we can recognize the Taylor series expansions for sine and cosine.

This particular differential equation is an example of the use of the method of Frobenius, formalized by

Frobenius General Rule I
If the indicial equation has **two distinct roots**, $c = \alpha, \beta$ ($\alpha < \beta$), **whose difference is an integer**, and one of the coefficients of x^k becomes indeterminate on putting $c = \alpha$, both solutions can be generated by putting $c = \alpha$ in the recurrence relation.

† In the usual way, we must show that (1.15) is true for $n = 0$ and that, when the value of a_{2n+1} is substituted into the recurrence relation, we obtain $a_{2(n+1)+1}$, as given by substituting $n + 1$ for n in (1.15).

In the above example the indicial equation was $c^2 - \frac{1}{4} = 0$, which has solutions $c = \pm\frac{1}{2}$, whose difference is an integer. The coefficient of x^{c+1} was $a_1\left\{(c+1)^2 - \frac{1}{4}\right\} = 0$. When we choose the lower of the two values $(c = -\frac{1}{2})$ this expression does not give us any information about the constant a_1, in other words a_1 is indeterminate.

1.3.2 The Roots of the Indicial Equation Differ by a Noninteger Quantity

We now consider the differential equation

$$2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0. \tag{1.16}$$

As before, let's assume that the solution can be written as the power series (1.8). As in the previous example, this can be differentiated and substituted into the equation to yield

$$2x(1-x)\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + (1-x)\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1}$$

$$+3\sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

The various terms can be multiplied out, which gives us

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)2x^{n+c-1} - \sum_{n=0}^{\infty} a_n(n+c)(n+c-1)2x^{n+c}$$

$$+\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} - \sum_{n=0}^{\infty} a_n(n+c)x^{n+c} + 3\sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

Collecting similar terms gives

$$\sum_{n=0}^{\infty} a_n\{2(n+c)(n+c-1) + (n+c)\}x^{n+c-1}$$

$$+\sum_{n=0}^{\infty} a_n\{3 - 2(n+c)(n+c-1) - (n+c)\}x^{n+c} = 0,$$

and hence

$$\sum_{n=0}^{\infty} a_n(n+c)(2n+2c-1)x^{n+c-1} + \sum_{n=0}^{\infty} a_n\{3 - (n+c)(2n+2c-1)\}x^{n+c} = 0.$$

We now extract the first term from the left hand summation so that both summations start with a term proportional to x^c. This gives

$$a_0 c(2c-1)x^{c-1} + \sum_{n=1}^{\infty} a_n(n+c)(2n+2c-1)x^{n+c-1}$$

$$+\sum_{n=0}^{\infty} a_n\{3-(n+c)(2n+2c-1)\}x^{n+c}=0.$$

We now let $m=n+1$ in the second summation, which then becomes

$$\sum_{m=1}^{\infty} a_{m-1}\{3-(m-1+c)(2(m-1)+2c-1)\}x^{m+c-1}.$$

We again note that m is merely a dummy variable which for ease we rewrite as n, which gives

$$a_0c(2c-1)x^{c-1}+\sum_{n=1}^{\infty} a_n(n+c)(2n+2c-1)x^{n+c-1}$$

$$+\sum_{n=1}^{\infty} a_{n-1}\left\{3-(n-1+c)(2n+2c-3)\right\}x^{n+c-1}=0.$$

Finally, we can combine the two summations to give

$$a_0c(2c-1)x^{c-1}$$

$$+\sum_{n=1}^{\infty}\{a_n(n+c)(2n+2c-1)+a_{n-1}\{3-(n-1+c)(2n+2c-3)\}\}x^{n+c-1}=0.$$

As in the previous example we can now consider the coefficients of successive powers of x. We start with the coefficient of x^{c-1}, which gives the indicial equation, $a_0c(2c-1)=0$. Since $a_0\neq 0$, this implies that $c=0$ or $c=\frac{1}{2}$. Notice that these roots do not differ by an integer. The general term in the summation shows that

$$a_n=a_{n-1}\left\{\frac{(n+c-1)(2n+2c-3)-3}{(n+c)(2n+2c-1)}\right\}\quad\text{for } n=1,2,\ldots\,. \tag{1.17}$$

We now need to solve this recurrence relation, considering each root of the indicial equation separately.

Case I: $c=0$

In this case, we can rewrite the recurrence relation (1.17) as

$$a_n=a_{n-1}\left\{\frac{(n-1)(2n-3)-3}{n(2n-1)}\right\}=a_{n-1}\left\{\frac{2n^2-5n}{n(2n-1)}\right\}=a_{n-1}\left(\frac{2n-5}{2n-1}\right).$$

We recall that this holds for $n\geqslant 1$, so we start with $n=1$, which yields

$$a_1=a_0\left(-\frac{3}{1}\right)=-3a_0.$$

For $n=2$

$$a_2=a_1\left(-\frac{1}{3}\right)=-3a_0\left(-\frac{1}{3}\right)=a_0,$$

where we have used the expression for a_1 in terms of a_0. Now progressing to $n = 3$, we have

$$a_3 = a_2 \left(\frac{1}{5}\right) = a_0 \frac{3}{5 \cdot 3},$$

and for $n = 4$,

$$a_4 = a_3 \left(\frac{3}{7}\right) = a_0 \frac{3}{7 \cdot 5}.$$

Finally, for $n = 5$ we have

$$a_5 = a_4 \left(\frac{5}{9}\right) = a_0 \frac{3}{9 \cdot 7}.$$

In general,

$$a_n = \frac{3a_0}{(2n-1)(2n-3)},$$

which again can be proved by induction. We conclude that one solution of the differential equation is

$$y = x^c \sum_{n=0}^{\infty} a_n x^n = x^0 \sum_{n=0}^{\infty} \frac{3a_0}{(2n-1)(2n-3)} x^n.$$

This can be tidied up by putting $3a_0 = A$, so that the solution is

$$y = A \sum_{n=0}^{\infty} \frac{x^n}{(2n-1)(2n-3)}. \tag{1.18}$$

Note that there is no obvious way of writing this solution in terms of elementary functions. In addition, a simple application of the ratio test shows that this power series is only convergent for $|x| \leqslant 1$, for reasons that we discuss below.

A simple MATLAB† function that evaluates (1.18) is

```
function frob = frob(x)
n = 100:-1:0; a = 1./(2*n-1)./(2*n-3);
frob = polyval(a,x);
```

which sums the first 100 terms of the series. The function `polyval` evaluates the polynomial formed by the first 100 terms in the sum (1.18) in an efficient manner. Figure 1.2 can then be produced using the command `ezplot(@frob,[-1,1])`.

Although we could now use the method of reduction of order, since we have constructed a solution, this would be very complicated. It is easier to consider the second root of the indicial equation.

† See Appendix 7 for a short introduction to MATLAB.

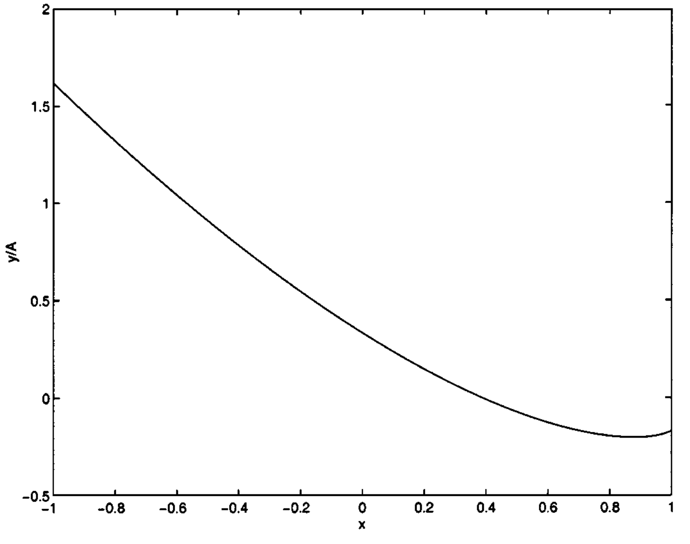

Fig. 1.2. The solution of (1.16) given by (1.18).

Case II: $c = \frac{1}{2}$

In this case, we simplify the recurrence relation (1.17) to give

$$a_n = a_{n-1}\left\{\frac{\left(n-\frac{1}{2}\right)(2n-2)-3}{\left(n+\frac{1}{2}\right)2n}\right\} = a_{n-1}\left(\frac{2n^2-3n-2}{2n^2+n}\right)$$

$$= a_{n-1}\left\{\frac{(2n+1)(n-2)}{n(2n+1)}\right\} = a_{n-1}\left(\frac{n-2}{n}\right).$$

We again recall that this relation holds for $n \geqslant 1$ and start with $n = 1$, which gives $a_1 = a_0(-1)$. Substituting $n = 2$ gives $a_2 = 0$ and, since all successive a_i will be written in terms of a_2, $a_i = 0$ for $i = 2, 3, \ldots$. The second solution of the equation is therefore $y = Bx^{1/2}(1-x)$. We can now use this simple solution in the reduction of order formula, (1.3), to determine an analytical formula for the first solution, (1.18). For example, for $0 \leqslant x \leqslant 1$, we find that (1.18) can be written as

$$y = -\frac{1}{6}A\left[3x - 2 + 3x^{1/2}(1-x)\log\left\{\frac{1+x^{1/2}}{(1-x)^{1/2}}\right\}\right].$$

This expression has a logarithmic singularity in its derivative at $x = 1$, which explains why the radius of convergence of the power series solution (1.18) is $|x| \leqslant 1$.

This differential equation is an example of the second major case of the method of Frobenius, formalized by

Frobenius General Rule II
If the indicial equation has **two distinct roots**, $c = \alpha, \beta$ ($\alpha < \beta$), **whose difference is** *not* **an integer**, the general solution of the equation is found by successively substituting $c = \alpha$ then $c = \beta$ into the general recurrence relation.

1.3.3 The Roots of the Indicial Equation are Equal

Let's try to determine the two solutions of the differential equation

$$x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0.$$

We substitute in the standard power series, (1.8), which gives

$$x\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + (1+x)\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1}$$

$$+2\sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

This can be simplified to give

$$\sum_{n=0}^{\infty} a_n(n+c)^2 x^{n+c-1} + \sum_{n=0}^{\infty} a_n(n+c+2)x^{n+c} = 0.$$

We can extract the first term from the left hand summation to give

$$a_0c^2x^{c-1} + \sum_{n=1}^{\infty} a_n(n+c)^2 x^{n+c-1} + \sum_{n=0}^{\infty} a_n(n+c+2)x^{n+c} = 0.$$

Now shifting the series using $m = n+1$ (and subsequently changing dummy variables from m to n) we have

$$a_0c^2x^{c-1} + \sum_{n=1}^{\infty}\{a_n(n+c)^2 + a_{n-1}(n+c+1)\}x^{n+c} = 0, \tag{1.19}$$

where we have combined the two summations. The indicial equation is $c^2 = 0$ which has a **double root** at $c = 0$. We know that there must be two solutions, but it appears that there is only one available to us. For the moment let's see how far we can get by setting $c = 0$. The recurrence relation is then

$$a_n = -a_{n-1}\frac{n+1}{n^2} \quad \text{for } n = 1, 2, \ldots .$$

When $n = 1$ we find that

$$a_1 = -a_0\frac{2}{1^2},$$

and with $n = 2$,

$$a_2 = -a_1 \frac{3}{2^2} = a_0 \frac{3 \cdot 2}{1^2 \cdot 2^2}.$$

Using $n = 3$ gives

$$a_3 = -a_2 \frac{4}{3^2} = -a_0 \frac{4 \cdot 3 \cdot 2}{1^2 \cdot 2^2 \cdot 3^2},$$

and we conclude that

$$a_n = (-1)^n \frac{(n+1)!}{(n!)^2} a_0 = (-1)^n \frac{n+1}{n!} a_0.$$

One solution is therefore

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{n!} x^n,$$

which can also be written as

$$y = a_0 \left\{ x \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right\}$$

$$= a_0 \left\{ -x \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m!} + e^{-x} \right\} = a_0 (1 - x) e^{-x}.$$

This solution is one that we could not have readily determined simply by inspection. We could now use the method of reduction of order to find the second solution, but we will proceed with the method of Frobenius so that we can see how it works in this case.

Consider (1.19), which we write out more fully as

$$x \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + 2y =$$
$$a_0 c^2 x^{c-1} + \sum_{n=1}^{\infty} \{a_n (n+c)^2 + a_{n-1}(n+c+1)\} x^{n+c} = 0.$$

The best we can do at this stage is to set $a_n(n+c)^2 + a_{n-1}(n+c+1) = 0$ for $n \geqslant 1$, as this gets rid of most of the terms. This gives us a_n as a function of c for $n \geqslant 1$, and leaves us with

$$x \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + 2y = a_0 c^2 x^{c-1}. \tag{1.20}$$

Let's now take a partial derivative with respect to c, where we regard y as a function of both x and c, making use of

$$\frac{d}{dx} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial c} \left(\frac{\partial y}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial c} \right).$$

This gives

$$x \frac{\partial^2}{\partial x^2} \left(\frac{\partial y}{\partial c} \right) + (1 + x) \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial c} \right) + 2 \left(\frac{\partial y}{\partial c} \right) = a_0 \frac{\partial}{\partial c} (c^2 x^{c-1}).$$

Notice that we have used the fact that a_0 is independent of c. We need to be careful when evaluating the right hand side of this expression. Differentiating using the product rule we have

$$\frac{\partial}{\partial c}(c^2 x^{c-1}) = c^2 \frac{\partial}{\partial c}(x^{c-1}) + x^{c-1}\frac{\partial}{\partial c}(c^2).$$

We rewrite x^{c-1} as $x^c\, x^{-1} = e^{c\log x}x^{-1}$, so that we have

$$\frac{\partial}{\partial c}(c^2 x^{c-1}) = c^2 \frac{\partial}{\partial c}(e^{c\log x}x^{-1}) + x^{c-1}\frac{\partial}{\partial c}(c^2).$$

Differentiating the exponential gives

$$\frac{\partial}{\partial c}(c^2 x^{c-1}) = c^2(\log x\, e^{c\log x})x^{-1} + x^{c-1}2c,$$

which we can tidy up to give

$$\frac{\partial}{\partial c}(c^2 x^{c-1}) = c^2 x^{c-1}\log x + x^{c-1}2c.$$

Substituting this form back into the differential equation gives

$$x\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial c}\right) + (1+x)\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}\right) + 2\frac{\partial y}{\partial c} = a_0\{c^2 x^{c-1}\log x + x^{c-1}2c\}.$$

Now letting $c \to 0$ gives

$$x\frac{\partial^2}{\partial x^2}\left.\frac{\partial y}{\partial c}\right|_{c=0} + (1+x)\frac{\partial}{\partial x}\left.\frac{\partial y}{\partial c}\right|_{c=0} + 2\left.\frac{\partial y}{\partial c}\right|_{c=0} = 0.$$

Notice that this procedure only works because (1.20) has a repeated root at $c = 0$. We conclude that $\left.\dfrac{\partial y}{\partial c}\right|_{c=0}$ is a second solution of our ordinary differential equation.

To construct this solution, we differentiate the power series (1.8) (carefully!) to give

$$\frac{\partial y}{\partial c} = x^c\sum_{n=0}^{\infty}\frac{da_n}{dc}x^n + \sum_{n=0}^{\infty}a_n x^n\, x^c\log x,$$

using a similar technique as before to deal with the differentiation of x^c with respect to c. Note that, although a_0 is not a function of c, the other coefficients are. Putting $c = 0$ gives

$$\left.\frac{\partial y}{\partial c}\right|_{c=0} = \sum_{n=0}^{\infty}\left.\frac{da_n}{dc}\right|_{c=0}x^n + \log x\sum_{n=0}^{\infty}\left.a_n\right|_{c=0}x^n.$$

We therefore need to determine $\left.\dfrac{da_n}{dc}\right|_{c=0}$. We begin with the recurrence relation, which is

$$a_n = -\frac{a_{n-1}(n+c+1)}{(n+c)^2}.$$

Starting with $n = 1$ we find that

$$a_1 = \frac{-a_0(c+2)}{(c+1)^2},$$

whilst for $n = 2$,

$$a_2 = \frac{-a_1(c+3)}{(c+2)^2},$$

and substituting for a_1 in terms of a_0 gives us

$$a_2 = \frac{a_0(c+2)(c+3)}{(c+1)^2(c+2)^2}.$$

This process can be continued to give

$$a_n = (-1)^n a_0 \frac{(c+2)(c+3)\dots(c+n+1)}{(c+1)^2(c+2)^2\dots(c+n)^2},$$

which we can write as

$$a_n = (-1)^n a_0 \frac{\prod_{j=1}^{n}(c+j+1)}{\left\{\prod_{j=1}^{n}(c+j)\right\}^2}.$$

We now take the logarithm of this expression, recalling that the logarithm of a product is the sum of the terms, which leads to

$$\begin{aligned}\log(a_n) &= \log((-1)^n a_0) + \log\left(\prod_{j=1}^{n}(c+j+1)\right) - 2\log\left(\prod_{j=1}^{n}(c+j)\right)\\ &= \log((-1)^n a_0) + \sum_{j=1}^{n}\log(c+j+1) - 2\sum_{j=1}^{n}\log(c+j).\end{aligned}$$

Now differentiating with respect to c gives

$$\frac{1}{a_n}\frac{da_n}{dc} = \sum_{j=1}^{n}\frac{1}{c+j+1} - 2\sum_{j=1}^{n}\frac{1}{c+j},$$

and setting $c = 0$ we have

$$\left.\left(\frac{1}{a_n}\frac{da_n}{dc}\right)\right|_{c=0} = \sum_{j=1}^{n}\frac{1}{j+1} - 2\sum_{j=1}^{n}\frac{1}{j}.$$

Since we know a_n when $c = 0$, we can write

$$\begin{aligned}\left.\frac{da_n}{dc}\right|_{c=0} &= (-1)^n a_0 \frac{\prod_{j=1}^{n}(j+1)}{\left(\prod_{j=1}^{n} j\right)^2}\left(\sum_{j=1}^{n}\frac{1}{j+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right),\\ &= (-1)^n a_0 \frac{(n+1)!}{(n!)^2}\left(\sum_{j=1}^{n+1}\frac{1}{j} - 1 - 2\sum_{j=1}^{n}\frac{1}{j}\right).\end{aligned}$$

In this expression, we have manipulated the products and written them as factorials, changed the first summation and removed the extra term that this incurs.